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ABSTRACT

The second volume of this proceedings contains the first portion of the research reports. Papers include: (1) "Learning Algebraic Strategies Using a Computerized Balance Model" (James Aczel); (2) "Children's Perception of Multiplicative Structure in Diagrams" (Bjornar Alseth); (3) "A Discussion of Different Approaches to Arithmetic Teaching" (Julia Anghileri); (4) "A Model for Analyzing the Transition to Formal Proofs in Geometry" (Ferdinando Arzarello, Chiara Micheletti, Federica Olivero, Ornella Robutti, and Domingo Paola); (5) "Dragging in Cabri and Modalities of Transition from Conjectures to Proofs in Geometry" (Ferdinando Arzarello, Chiara Micheletti, Federica Olivero, Ornella Robutti, Domingo Paola, and Gemma Gallino); (6) "The Co-Construction of Mathematical Knowledge: The Effect of an Intervention Program on Primary Pupils' Attainment" (Mike Askew, Tamara Bibby, and Margaret Brown); (7) "Dialectical Proof: Should We Teach It to Physics Students?" (Roberto Ribeiro Baldino); (8) "Lacan and the School's Credit System" (Roberto Ribeiro Baldino and Tania Cristina Baptista Cabral); (9) "Which Is the Shape of an Ellipse? A Cognitive Analysis of an Historical Debate" (Maria G. Bartolini Bussi and Maria Alessandra Mariotti); (10) "Children's Understanding of the Decimal Numbers through the Use of the Ruler" (Milena Basso, Cinzia Bonotto, and Paolo Sozio); (11) "Construction of Multiplicative Abstract Schema for Decimal-Number Numeration" (Annette R. Baturo and Tom J. Cooper); (12) "Classroom-Based Research to Evaluate a Model Staff Development Project in Mathematics" (Joanne Rossi Becker and Barbara J. Pence); (13) "Some Misconceptions Underlying First-Year Students' Understanding of 'Average Rate' and of 'Average Value'" (Jan Bezuidenhout, Piet Human, and Alwyn Olivier); (14) "Operable Definitions in Advanced Mathematics: The Case of the Least Upper Bound" (Liz Bills and David Tall); (15) "Beyond 'Street' Mathematics: The Challenge of Situated Cognition" (Jo Boaler); (16) "The 'Voices and Echoes Game' and the Interiorization of Crucial Aspects of Theoretical Knowledge in a Vygotskian Perspective: Ongoing

Research" (Paolo Boero, Bettina Pedemonte, Elisabetta Robotti, and Giampaolo Chiappini); (17) "Children's Construction of Initial Fraction Concepts" (George Booker); (18) "Graphing Calculators and Reorganization of Thinking: The Transition from Functions to Derivative" (Marcelo C. Borba and Monica E. Villarreal); (19) "Pre-Algebra: A Cognitive Perspective" (G.M. Boulton-Lewis, T. Cooper, B. Atweh, H. Pillay, and L. Wilss); (20) "The Right Baggage?" (Mary Briggs); (21) "Learner-Centered Teaching and Possibilities for Learning in South African Mathematics Classrooms" (Karin Brodie); (22) "Researching Transition in Mathematical Learning" (Tony Brown, Frank Eade, and Dave Wilson); (23) "Metaphor as Tool in Facilitating Preservice Teacher Development in Mathematical Problem Solving" (Olive Chapman); (24) "Restructuring Conceptual and Procedural Knowledge for Problem Representation" (Mohan Chinnappan); (25) "The Structure of Students' Beliefs towards the Teaching of Mathematics: Proposing and Testing a Structural Model" (Constantinos Christou and George N. Philippou); (26) "Abstract Schema versus Computational Proficiency in Percent Problem Solving" (Tom J. Cooper, Annette R. Baturo, and Shelley Dole); (27) "Implicit Cognitive Work in Putting Word Problems into Equation Form" (Anibal Cortes); (28) "Three Sides Equal Means It Is Not Isosceles" (Penelope Currie and John Pegg); (29) "Making Sense of Sine and Cosine Functions through Alternative Approaches: Computer and 'Experimental World' Contexts" (Nielce Lobo Da Costa and Sandra Magina); (30) "Teacher and Students' Flexible Thinking in Mathematics: Some Relations" (Maria Manuela M.S. David and Maria da Penha Lopes); (31) "The Influence of Metacognitive and Visual Scaffolds on the Predominance of the Linear Model" (Dirk De Bock, Lieven Verschaffel, and Dirk Janssens); (32) "To Teach Definitions in Geometry or Teach To Define?" (Michael De Villiers); (33) "Student thinking about Models of Growth and Decay" (Helen M. Doerr); (34) "Analysis of a Long Term Construction of the Angle Concept in the Field of Experience of Sunshadows" (Nadia Douek); (35) "On Verbal Addition and Subtraction in Mozambican Bantu Languages" (Jan Draisma); (36) "Teachers' Beliefs and the 'Problem' of the Social" (Paula Ensor); (37) "From Number Patterns to Algebra: A Cognitive Reflection on a Cape Flats Experience" (Clyde B.A. Felix); (38) "Affective Dimensions and Tertiary Mathematics Students" (Helen J. Forgasz and Gilah C. Leder); (39) "Social Class Inequalities in Mathematics Achievement: A Multilevel Analysis of TIMSS South Africa Data" (George Frempong); (40) "Context Influence on Mathematical Reasoning" (Fulvia Furinghetti and Domingo Paola); (41) "What Do They Really Think? What Students Think about the Median and Bisector of an Angle in the Triangle, What They Say and What Their Teachers Know about It" (Hagar Gal); (42) "Levels of Generalization in Linear Patterns" (Juan Antonio Garcia-Cruz and Antonio Martinon); (43) "The Evolution of Pupils' Ideas of Construction and Proof Using Hand-Held Dynamic Geometry Technology" (John Gardiner and Brian Hudson); and (44) "Cognitive Unity of Theorems and Difficulty of Proof" (Rosella Garuti, Paolo Boero, and Enrica Lemut). (ASK)

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Volume 2

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PME 22
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Editors: Alwyn Olivier and Karen Newstead

University of Stellenbosch
Stellenbosch, South Africa
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LEARNING ALGEBRAIC STRATEGIES USING A COMPUTERISED BALANCE MODEL

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A Popperian analysis of the research literature on the learning of algebra has yielded an interactive, game-like, computer-based balance model of the simple linear equation, that also incorporates word problems solvable by formulating such equations. A study is described that explores the potential of this software to improve the algebraic knowledge of 10 to 15-year old children. Questions are raised about the transferability of knowledge between types of algebra problems.

Some difficulties with the balance model

Lins (1992) describes the balance model as "one of the most popular didactic artefacts used to teach the solution of linear equations" (p. 208). Yet it cannot easily handle negative signs and negative numbers; and decimal coefficients can be difficult to visualise. Such restrictions "lead to major cognitive difficulties for students in the long run" (Linchevski & Herscovics, 1996).

But it is much criticised on a variety of other grounds in several research studies. Schliemann et al. (1992) found that children rarely use a spontaneous cancellation strategy to find unknown weights on physical balance scales; Dickson (1989) found a balance model did not lead to the desired learning; while Filloy & Rojano (1989) found deleterious effects, and the links between operations on the model and operations on the equation were often not made; and Lins concluded that the balance model is "inadequate not only for quickly becoming a complex net of what are in effect different models, but also for not fostering a frame of mind adequate for the development of an *algebraic mode of thinking*." (p. 209). Criticisms have also been made that such concrete materials encourage at best the use of letters as unknowns rather than as variables; and at worst as standing for objects rather than for numbers.

Moreover, it is not clear how the model might provide a purpose for *conventional* algebraic representation. Nor is it clear in a balance model whether an algebraic strategy is necessarily more attractive than numerical trial-and-improvement, a whole-part sharing strategy or inverse operations. So can the balance model be useful at all in helping students to appreciate the power, beauty and challenge of algebra?

The Popperian Psychological Perspective

A distinctive psychological perspective has been developed for the study of the mind in an educational setting, inspired by Popper (e.g.1968). It focuses on students' *strategic theories* and *concerns*.

Many theories can be seen as elementary strategies. For example, each of Küchemann's (1981) letter interpretations enables certain algebra problems to be tackled with varying degrees of success. The balance model provides a strategy of simplification while maintaining an equality.

From this perspective, learning is seen as creative, strategic, trial-and-error-elimination of theories in response to concerns. By "concerns", I mean problems of special interest to an individual, including desires, motivations and fears. The myriad explications of cognitive processes (such as the interplay of innate faculties, mental representations, modes of thought and gestalt) can be reinterpreted in terms of World 3 objects such as theories and concerns.

In the case of understanding and using equations, the perspective would characterise this knowledge in terms of strategic theories; and ask what problems these theories solve - such as how to represent a situation using algebraic notation; or how to simplify by operating on both sides of an equality. Could the balance model be adapted to make these problems into concerns? The study described below attempts to check this, by testing to see if a computerised version of the model makes the use of an algebraic strategy more attractive than other strategies.

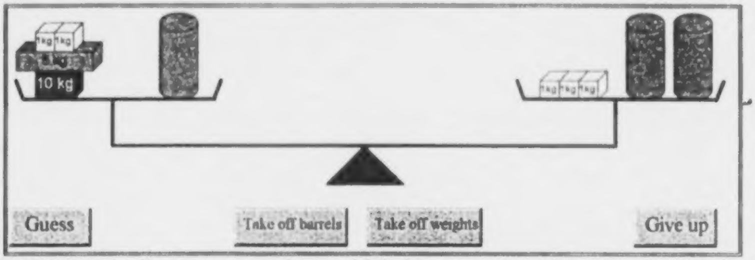
But are there "deeper" insights that have to be obtained in learning mathematics than the gaining of merely operational knowledge (e.g. Sierpinska, 1994)? The relationship between the "meanings" that students have - for letters, expressions, equations and operations - and the cognitive demands of using and solving equations can be questioned. It could be argued (given the Popperian view of the products of understanding as a succession of theories attempting to solve a problem) that such interpretations, images, concepts and meanings are not fundamental insights into students' cognitive processes, habits and resources; but rather the theoretical by-products of engagement with past problems (especially ones solved using arithmetic or proportion). Such "decontextualised theories" are valuable in that they may give insight into concerns, experiences and consequent rationalisations; and therefore perhaps may help teachers and researchers to conjecture students' theories in a given problem situation, or potentially act as strategies in another problem situation. They are not, however, indicative of an over-arching or underlying "conception" or cognitive structure (cf. Linchevski & Herscovics, 1996). The study therefore checks whether any decontextualised theories improve as a result of using the computerised balance model; i.e. if students learned something over and above that ostensibly involved in using equations to find an unknown number in a situation.

A Computerised Balance Model

One issue that has guided the computerisation of the model is whether it is more productive to aim to help students to represent situations using conventional algebraic notation; or to represent less conventionally but transform more easily. For example, students could be helped to represent conventionally by reflecting on the way symbols are used; by using syncopated language; by formalising trial-and-error; by employing a mathematics machine like that of Booth (1984); or by using computer algebra systems such as Derive and MathCad. Less conventional representations could include spreadsheets and computer languages (see Kieran, 1992).

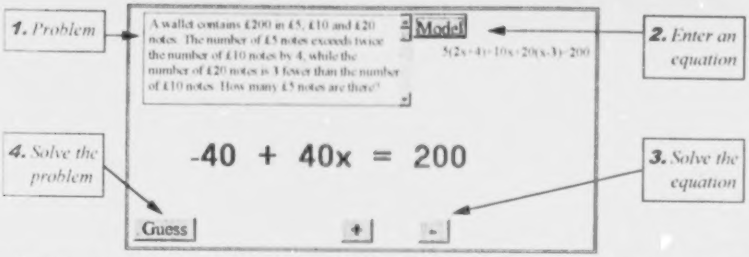
The conventional approach has to ensure that students realise that formal operations aim to simplify matters rather than constituting an end in themselves; whereas with the unconventional approach, very few activities are found in the research literature that involve equations as opposed to expressions or functions. Should simplification of expressions therefore precede solution of equations?

If it is not so much the act of *representing as an equation* that makes an algebraic strategy attractive, but the associated *simplification of the problem*, perhaps students could transform a given representation that gradually becomes conventional. The program - called EQUATION - therefore starts with randomly-generated balance puzzles that are initially accessible to informal strategies such as guessing or whole-part sharing; but as the puzzles get harder, such strategies are harder to implement. A simplification strategy (in the form of "Take off" buttons) should thus become more attractive. The de facto separation of operation choice from operation execution not only obviates the need for arithmetic, but enables students to focus on simplification decisions, and thereby improve their strategies without an initial requirement for accurate theories of operations on expressions.



EQUATION initially presents students with balance puzzles of the form $E + b = F$ and $Kb = E$, where E and F are weights, b represents the weight of a barrel, K is the number of barrels (less than 5 at this level). Level 2 puzzles are of the two-step form $E + Kb = F$ and (potentially across the "cut" of Filloy & Rojano) $E + Kb = (K + 1)b$. Puzzles on Levels 3-5 are of the form $E + Kb = F + (K+1)b$, $E + Kb = F + (K+2)b$ and $E + Kb = F + Lb$ respectively. Until this point, although division may be necessary, the answer is still ensured to be a whole number. On Level 6, the ⧄ key can be used to deal with fractional answers.

Algebraic notation is now introduced piecemeal. On Level 7, the multiple weight pictures are replaced by a single weight picture. On Level 8, the barrels are labelled with the letter 'b', and the two "Take Off" buttons are replaced by a single ⊟ button. On Levels 9 and 10, the balance pictures are replaced by symbolic notation. Level 11 equations involve negative answers, breaking with the balancing context. Level 12 equations involve negative signs and a ⊞ button. Future features could include a tilting balance, subtraction via dragging, simultaneous equations, rearrangement and quadratics.



A wallet contains £200 in £5, £10 and £20 notes. The number of £5 notes exceeds twice the number of £10 notes by 4, while the number of £20 notes is 3 fewer than the number of £10 notes. How many £5 notes are there?

$5(2x+4)+10x+20(x-3)=200$

$-40 + 40x = 200$

The final levels present students with word problems (such as those in Lins, 1992) that can be tackled using an optional "Model" button to enter an equation. On Level 13, each problem is a description of a balance puzzle. Level 14 problems may include negative signs; Level 15 involves combining ratios; and Level 16 requires expressions.

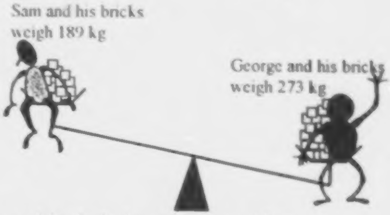
Exploring the Computerised Balance Model

The empirical work described here aimed to analyse how the algebraic knowledge of a class of 22 students aged 14-15 could improve as a result of EQUATION; in particular whether an algebraic strategy (such as representing a situation using conventional algebraic notation, and simplifying by operating on both sides of an equality) was chosen in preference to trial-and-improvement or whole-part sharing.

The students were given a pre-test based on items in the research literature, especially Booth (1984). These items were divided into 4 problem types. *Representation* items included "C1(i) Find the area of this shape [rectangle shown with sides *n* and *m*]"; "C2(iii) Find the perimeter of this shape [partial diagram given of a shape declared to have *n* sides, each of length 2]"; "C4 Describe a situation in which $x = 4c$ could help you or tell you something."; "C5 Blue pencils cost 5p each and red pencils cost 6p each. I buy some blue and some red pencils and altogether it costs me 90p. If b is the number of blue pencils bought and if r is the number of red pencils bought, what can you write down about b and r?"; and the student-professor problem (C6). *Transformation* items included "D1(ii) Add 4 onto $3n$"; "D3(ii) Solve $5x + 4 = 4x - 31$"; "D4(iv) Write $2a + 5b + a$ more simply if possible"; and "D5 When is $L + M + N = L + P + N$ true? [Always, Sometimes, Never]".

Among the *Modelling* items was the seesaw problem adapted from Lins (1992).

Sam throws away some bricks and George throws away four times as many. Now they are balanced. How much weight did Sam throw away?


Sam and his bricks weigh 189 kg
George and his bricks weigh 273 kg

The fourth problem type was labelled *Patterns* and included a question with a sequence of piles of matches (4, 7, 10, 13...). A1(ii) asked for the number of matches in the 100th pile; and A1(iii) for the number of the pile with 568 matches.

The class also completed a questionnaire about their views of algebra, including "Why do you think some people find equations difficult?"; "How confident are you about algebra?"; "Have you ever made up an equation?"; and "If you had to explain to somebody younger what sort of thing an equation is, how would you describe it?". Two pairs of students were also given a semi-structured interview lasting about 30 minutes, in which their responses were probed more closely.

The class then used EQUATION for two hour-long lessons, mostly in pairs. One advantage of the Popperian perspective is that it should be possible to identify students' strategic theories and concerns not only through their responses to written items, but also in medias res, when learning is taking place. The program therefore recorded each pair's inputs to the computer in a log. This can be re-played on-screen or printed out; and so the research is not limited by the number of video cameras or tape recorders available. The conversations of the students who were interviewed were also audio-recorded. This data was collected to try to relate any learning to the problems contained in the software.

Finally, the class was given a post-test and the students who had been interviewed were re-interviewed. This was to detect prima facie evidence of improvement in theories and concerns. A parallel class was given both tests as a control, but did not work on algebra between the two tests.

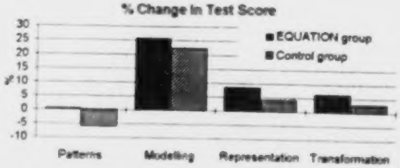
Some Test Results

The class had already been taught much algebra - including simplifying expressions, solving linear, quadratic and simultaneous equations, and functions. Yet the tests showed that many of the students struggled with algebra, and the questionnaires suggested that many saw algebra as a pointless ritual. The EQUATION group improved significantly in terms of raw score (p=0.0064 using a t-test); the control group did not. But the program is certainly no panacea: the post-test results for the EQUATION group are not always significantly higher than the control group.

	EQUATION Group					**Control Group**				
Item	**pre**	**post**	**imp**	**wor**	**p**	**pre**	**post**	**imp**	**wor**	**p**
C1(i)	82	86	2/4	1/18	0.288	92	96	2/2	1/22	0.287
C2(iii)	50	77	8/11	2/11	0.028	50	67	6/12	2/12	0.081
C4	9	36	7/20	1/2	0.015	58	38	2/10	7/14	0.048
C5	9	27	4/20	0/2	0.021	0	8	2/24	0/0	0.081
C6	45	55	6/12	4/10	0.270	38	33	6/15	7/9	0.394
D1(ii)	45	68	6/12	1/10	0.028	58	67	5/10	3/14	0.246
D3(ii)	18	45	8/18	2/4	0.028	4	8	2/23	1/1	0.287
D4(iv)	77	91	3/5	0/17	0.041	83	79	3/4	4/20	0.357
D5	27	59	7/16	0/6	<0.001	42	50	5/14	3/10	0.246
seesaw	23	59	10/17	2/5	0.009	29	42	3/17	0/7	0.041
A1(ii)	45	55	5/12	3/10	0.246	54	71	7/11	3/13	0.106
A1(iii)	50	50	5/11	5/11	0.500	54	46	2/11	4/13	0.213

NB **pre** and **post** are facilities (%), **imp** = {number of students who improved} / {number who could have improved}, **wor** = {number who worsened} / {number who could have worsened}

Equation-solving improved, as one might expect. But more interestingly, students improved in modelling whether or not they used EQUATION. Looking at the strategies used, an increased concern to use an algebraic strategy in the EQUATION group contrasted with an increased success with trial-and-improvement in the control group.



The EQUATION group used the following strategies in the seesaw problem:

Pre	-	✓T	-	?	?	-	-	W	✓T	W	✓?	W	-	-	W	-	W	✓T	✓?	-	W	-
Post	W	✓A	A	?	-	W	W	A	?	✓?	✓?	✓?	✓A	✓?	✓A	✓?	✓A	A	✓?	✓A	✓T	✓A

A = Algebraic, T = Trial-and-error, W = Whole-part, ? = unknown, - = question left blank, ✓ = correct

Nevertheless, transfer between modelling problems appears to be non-trivial: algebraic strategies were not used in A1(iii).

Intriguingly, the alleged dangers associated with *b* being used to stand for "barrel" rather than "weight of a barrel" do not appear to be reflected in items C6 and D1(ii).

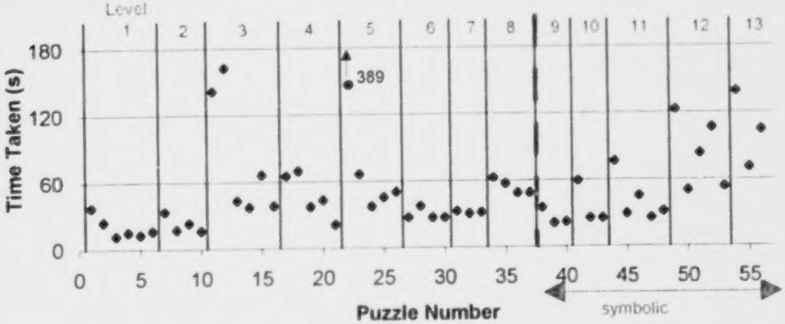
Two Students' Interactions with EQUATION

The class logs have not yet been fully analysed, but preliminary analysis suggests that they show the active creation and improvement of strategic theories. This is especially convincing when listening to the audio-tape of the students' conversations while the program replays on the screen what the students saw and did.

Rebecca and Nicola, working together, solve puzzles on the first two levels quite easily. They then simplify a puzzle from 19kg + 4 barrels = 12kg + 5 barrels to 19kg + 1 barrels = 12kg + 2 barrels, but it is not until another student suggests it that Rebecca says in confident tones "Yeah, take off one barrel.". There's a pause and then she asks him "What... have you done this one before?", suggesting that she still doesn't appreciate the need for the barrel to go. But when the new picture appears, she exclaims, "Ah that's obvious now" and quickly gets the answer. The question is now - can Rebecca modify the theory to take off in one go as many barrels as there are on the side with the smaller number of barrels? A few similar puzzles later, and she can.

Nicola, however, is still struggling to grasp Rebecca's strategy. For example, faced with $49 = 19 + 2b$, Rebecca has no hesitation in doing $(49 - 19) \div 2$; but Nicola asks "Why are you minusing it?". Rebecca's reply - "Because then you get a balanced equation and then you just divide it by the last two barrels left." - indicates that although Rebecca has a good theory for solving the puzzle, her rationale does not involve the idea of simplifying a situation. In any case, Nicola is quite happy using subtraction on the very next puzzle - further evidence that grasping a theory does not always depend on having a coherent rationale for it. It then takes them some time to work out that they can take off as much weight as there is on the side with the smallest number of kg displayed. In other words, the theory for barrels does not automatically get employed for weights.

The move from pictures to symbols on Level 9 does not cause comment - far from confusing them, they continue with their strategy and in fact are even quicker than with the pictures because they no longer have to count barrels. The following graph illustrates this remarkable result, and also indicates the improvement in strategic theories occurring in response to changes in the nature of the problem at each level:



The break with the balance model on Levels 11 and 12 was less traumatic than might be the case with paper-based exercises: the following puzzle log shows the sort of exploration of algebraic form that enabled students to develop strategic theories to cope with negative answers and negative signs:

#	Action	Resulting Equation	Time (s)
		$-22 - 14b = -12 + 4b$	
1	Subtract 12	$-34 - 14b = -24 + 4b$	13
2	Subtract 4*b*	$-34 - 18b = -24$	12
3	Add 24	$-10 - 18b = 0$	13
4	Add 10	$-18b = 10$	13
5	Guess -18/10		24
6	Guess 10/-18	Correct	10

Note how the feedback to Action 1 allows them to debug their strategy. Nicola says "You should've *plussed* 12.". Rebecca isn't convinced, until she tries it.

Rebecca and Nicola seem well aware that choosing an unhelpful operation is not fatal to the solution process. They can experiment with algebra, try hunches and make mistakes without having to start from scratch. The evident satisfaction derived from this success kept the whole class on task for an hour's lesson. However, the above graph suggests that Rebecca and Nicola need more practice in equations with negative signs, to consolidate the variety of possible permutations. Moreover, the transfer from computer-based solution to pencil-and-paper should not be taken for granted.

In retrospect, the modelling problems could have been graduated better - being asked to find a quantity that was not the obvious choice to be represented by a letter was a bigger hurdle than anticipated. Nevertheless, it was amazing not so much that students chose to formulate their own equations and that they were able to; but that once the equation appeared on the screen, students said things like "Ah, now I can do it!" and "It's easy now!". In other words, the equation had become for them a powerful problem-solving tool that they were confident about using. Enjoyment in using algebra was, for many of the students in the study, a new experience.

Conclusions

This study suggests that the objections that might be applied to a physical balance or a balance picture in a textbook do not apply to the interactive, game-like, computer-based balance model in EQUATION.

Did any decontextualised theories improve as a result of using EQUATION? Prima facie evidence (with some provisos) of improved theories for representation and transformation items would suggest so, even though EQUATION does not instruct students in tackling such items. It can be argued that this is because the strategic theories developed when using the program are robust enough for use in other contexts. The hypothesis that those who struggle with algorithms require explicit consideration of meaning in a variable-centred approach to algebra would appear to be challenged by the apparently greater understanding of the role of algebraic representation demonstrated by these students. However, test conditions, students' informal discussions between the tests, and the excitement of being involved in research might prove to be better explanations for statistical significance than EQUATION itself. Moreover, there are no guarantees that these improvements are sustained over long time-scales. Therefore, given that this empirical work was primarily to illustrate a theoretical analysis of learning processes, rather than to achieve a large, rigidly controlled experiment with random sampling and allocation, further research is required to substantiate such claims about effects.

Acknowledgement

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CHILDREN'S PERCEPTION OF MULTIPLICATIVE STRUCTURE IN DIAGRAMS

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Abstract

This paper reports from a larger study of children's perception of multiplicative structure and how this develops over a period of time. In the reported study, 16 8-year-olds were asked to consider whether various drawings were appropriate and helpful or not for someone trying to solve a given problem. The students accepted drawings that showed the correct number of objects no matter how these objects were arranged. It was not necessary that the drawing reflected the mathematical structure in the problem.

Background

Children's use of diagrams and drawings while performing mathematics has interested many researchers in mathematics education. The research has mainly focused on the use of diagrams or drawings in solving mathematical problems. In a meta-analysis Hembree (1992) finds that "draw a diagram" is the strategy that has been most successful in experiments conducted to improve students' ability to solve mathematical problems. In a study of students' use of diagrams, Lopez-Real and Veloo (1993) presented various text problems to 96 students in grade 5 and 6. Out of a total of 693 responses, only 5% used diagrams. They got 126 responses which were considered wrong or incomplete. In such cases, the interviewer suggested that the student made a diagram. In 107 of these cases, the student was able to do so, which led to a correct solution in 41 cases. In 78 of the remaining 86 cases without a correct solution, the interviewer presented a ready-made diagram. With this aid, the students were able to find a solution in 52 of the 78 cases. From this Lopez-Real and Veloo conclude that 1) students do not use diagrams very often, 2) many students are able to draw and use a diagram to solve a problem when they are stuck, and 3) students are able to interpret and use a diagram to solve a problem even though they cannot produce one themselves.

In addition to being used as a problem solving aid, diagrams may play a significant role in students' development of mathematical concepts. This role of diagrams has not been researched to the same extent, even though several researchers include a visual component in their theoretical framework for describing students' mathematical conceptions. Goldin (1992) puts emphasis on "imagery" in his attempt to build a unified model for describing conceptual systems. One central element in this model is a person's internal representations of a mathematical concept, where "*imagistic systems*" is one important aspect. The representations within this category will be more or less visual since the category also includes auditorial and tactile systems. Gutstein and Romberg (1995) discuss the use of diagrams and other mediative representations in the teaching of addition and subtraction. In their review of research in this area, they find two main directions: one where it is supposed best to let the children develop their own diagrams and the other where the children are taught specific diagrams that are supposed to be useful. The research in both of these dimensions shows good results for the participants, and Gutstein and Romberg conclude that it is probably best *both* to let the children invent their own diagrams (and procedures) *and* to teach them directly how to use diagrams (and procedures) that are connected "*to students' prior knowledge at every point in the process*" (p. 317).

Reynolds and Wheatly (1992) present interviews of four grade 4 students and one six-year-old. During the interviews the students are given some unfamiliar mathematical problems and encouraged to express how they "see" the problem and eventually use pencil and paper as an to find a solution. During the interviews Reynolds and Wheatly find that the students mathematical thinking is heavily based on images: "*we believe that meaningful mathematics is image-based*" (p. 248), but also that: "*images may not be used when students perform prescribed computational methods*" (p. 248). There has been conducted an extensive amount of research on students' algorithms for solving problems with additive or multiplicative structures (see Fuson, 1992, and Greer, 1992, for reviews on this research). The study of Reynolds and Wheatly (1992) suggests that it can be of interest to study children's mental images while working with such problems as well.

Thomas, Mulligan and Goldin (1994) describe parts of children's conceptual understanding of the number system by analysing children's visual representations of the numbers from 1 to 100. The children in the study were asked to close their eyes and to imagine the numbers from 1 to 100. Then they were asked to draw the pictures they saw in their mind. The responses show a great diversity of representations that to a varying degree embody the structural aspects of our number system. Thomas et al. conclude that "*the children's internal representations of numbers are highly imagistic*" (p. 7). Another result of their research is that the children's drawings to a small degree reflect the structures in the number system, e.g. how it is built up by units of ten.

The distinction between diagram as a part of ones mathematical conceptions and as a problem solving device is not clear. As Nunokawa (1994, p. 34) points out:

> *In the problem-solving process, the solver gives a certain structure to the problem situation in which the question is asked. This structure consists of those elements the solver recognises in the situation, the relationships (s)he establishes among these elements, and the senses (s)he gives to the elements or the relationships.*

The structure the problem solver is working within, is not inherited directly from the problem situation, but imposed on the situation by the problem solver. Therefore a student's use of diagram in a specific problem solving situation will depend on his/her knowledge of the mathematical concepts involved and on the application of diagrams as a problem solving tool in general. In the study by Lopez-Real and Veloo (1993), students in 85 out of 126 cases were not able to produce a helpful diagram when this was suggested. This might be because of a lack of knowledge about the use of diagrams as a problem solving tool, or because, as Nunokawa suggests, the mathematical structure in a problem situation is not sufficiently recognised for the production of a diagram. The student's conceptual knowledge may not contain the elements necessary for discerning the mathematical structure embedded in the situation.

This connects diagrams with a student's mathematical concept knowledge. On the other hand, there are researchers who separates these aspects. Fischbein (1987, p. 158) claims that: "*A diagram is necessarily a post-conceptual structure*". The aim of the research I'm undertaking is therefore to investigate the relationship between

- students' perception of mathematical structure in a problem situation and their use of diagrams as representations of this structure and
- their content knowledge regarding that structure and the development of this knowledge.

This paper presents a part of these investigations.

Method

The results reported here are based on interviews of sixteen 8-year-olds who had just started their second year of schooling. None of the students had received any formal education in multiplication prior to the test. Half of the students came from a small town school, the other half came from a class in a school in an upper-class area in a big town. Both samples showed a wide range of abilities. The children were interviewed as part of a longitudinal study. The interviews were based on selected tasks on multiplication and place value. All tasks contained a visual component. This was insured by asking the children to produce drawings or diagrams as a response to the tasks or by asking the children to comment on different drawings/diagrams. The responses to two of the tasks are presented here.

Each of the two tasks consisted of a problem statement and 11 drawings or diagrams that might be helpful to a student trying to solve the problem. In the interview, the problem statements were read by the interviewer, several times if necessary. Then the drawings were presented one-by-one, and the students were asked whether the drawing was or was not appropriate to the problem and helpful for someone trying to solve it. The drawings were made so that they did reflect the structure in the problem situations in different ways: in groups, along the number line, in arrays. For each such drawing there were made drawings that looked quite similar, but that did not reflect the structure.

Discussion and analysis

The first item was: *In a classroom there are 5 tables. There are 3 students seated at each table. How many students are there in the classroom?*

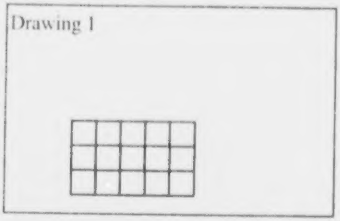
This item was solved correctly by all students except one. Some students found the correct answer almost immediately, while others solved it after seeing the first drawing. One student found the correct answer after seeing the second drawing, but he remained quite uncertain about the correctness of his answer all through the session.

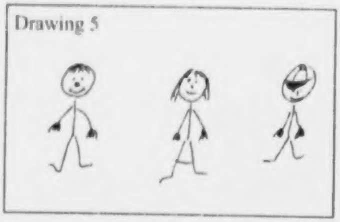
When asked if a particular drawing was appropriate to the task, the students usually answered "Yes". Table 1 shows the percentages of students answering "Yes", "No" or "Uncertain/Yes and No/Don't know".

Drawing	1	2	3	4	5	6	7	8	9	10	11
Yes	38	69	56	69	44	50	81	44	69	50	75
Uncert.	19	19	19	31	6	25	6	38	25	31	25
No	44	13	25	0	50	25	13	19	6	19	0

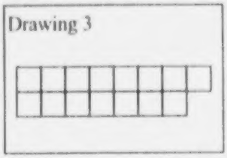
Table 1. Percentages of students considering the drawings 1 to 11 as appropriate to task 1.

Every drawing presented was considered to be appropriate and helpful to the task by more than one third of the students. Only for drawing 1 and 5 there were more *No's* than *Yes's*.


Drawing 1

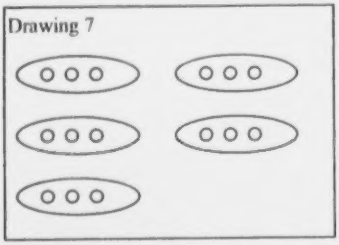

Drawing 5

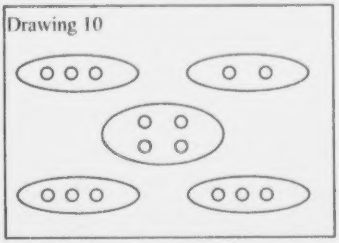
The students who did not accept drawing 1 as helpful, did so for mainly two reasons. Firstly because the squares were interpreted as the tables in the classroom. This is what Hughes calls a pictographic representation (Hughes, 1986, p. 57), a representation that incorporates both the shape and the position of the objects as well as the number of objects. These students counted in threes e.g. while tapping three times for each square. They therefore concluded that there are too many squares in the drawing. The second reason for not accepting this drawing, came from students who counted the intersections of the lines in drawing 1. This way they got 4x6 or 4x5 points which they regarded as too many.



Of the 6 students who answered "Yes" to the drawing 1, 5 answered "Yes" to drawing 3 as well. The 6th student hesitated and did not give an answer to the 3rd drawing. The students who thought drawing 1 was appropriate and helpful for this problem, seemed to do so because of the drawing's representation of the 15 objects in the drawing. That the drawing also reflects the 3 times 5 multiplicative structure seemed to be of less importance.

Similar pattern can be seen by comparing the responses to drawings 7 and 10:





13 students answered "Yes" and 1 student was uncertain to drawing 7, while the corresponding numbers for drawing 10 were 8 and 5. If a student solved this problem entirely by counting and (s)he had already figured out that the answer is 15, drawing 10 is as acceptable as drawing 7. This is probably the reason why half the students found drawing no. 10 helpful. Both drawings helped them do the counting something that confirmed their calculated answer. For these students the number of objects in the drawing seemed to be the most important criteria, the way the objects were organised and the relationships between the objects in the drawing played a less important role. The student did not relate the drawings to the multiplicative structure of the question. This assertion is confirmed by the fact that some of the students accepted drawing no. 5, a picture of 3 children. Even though most students did not find this drawing helpful, 7 students did. All of these used counting as their main strategy both for solving the actual problem and for judging whether the drawings were helpful or not. As commented by a girl who found the drawing of the 3 children helpful: "It's OK, it's 15, I counted several times."

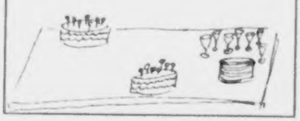
Item 2 was: *Joe bought 4 cookies and he paid 5 kroner for each cookie. How much did he pay?*

All students found a solution to this task before seeing any drawing, using counting or additional facts (e.g. "5+5=10 and 10+10=20"). Some of the students modified a wrong initial solution after seeing the first drawing. Table 2 shows the percentages of students answering "Yes", "No" or "Uncertain/Yes and No/Don't know" to this task.

Drawing	1	2	3	4	5	6	7	8	9	10	11
Yes	75	94	94	81	44	94	94	88	75	100	69
Uncert.	19	6	6	6	0	6	0	6	6	0	19
No	6	0	0	13	56	0	6	6	19	0	13

Table 2. Percentages of students considering the drawings 1 to 11 as appropriate to task 2.

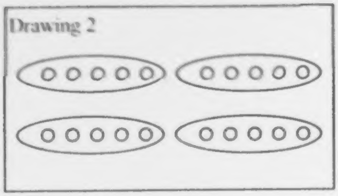
The number of drawings considered helpful is higher and there are fewer "Uncertain" than in task 1. One reason for this may be found in that more students solved the task correctly before starting judging the drawings. The only drawing that was rejected by more than half of the students was no. 5:

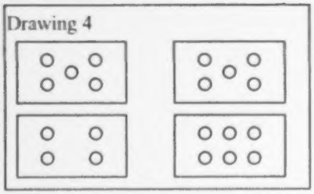


Drawing 5

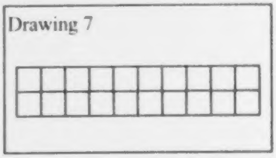
But still, 7 students found this drawing appropriate and helpful. Most of them counted, e.g. the number of candles, the number of plates and/or the number of glasses. If they got "20" one way or another, they concluded that the drawing was helpful. It should be noted that of the 7 students who accepted drawing 5, 4 did accept *all* the eleven drawings and 2 accepted all drawings in the first task as well. This implies that their reason for accepting can both be a reliance on a counting strategy and a wish to please the interviewer. They might have interpreted the instruction, not as "is this drawing helpful?", but as "is there a way that this drawing might be helpful?" which is easier to agree upon.

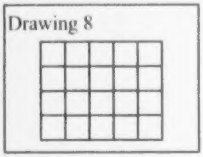
Acceptance of drawings which do not resemble the multiplicative structure in the problem, as found in task 1, was also found in task 2. The following drawings, no. 2 and 4, were both accepted by almost all students.


Drawing 2


Drawing 4

The same goes for drawings 7 and 8:


Drawing 7


Drawing 8

While drawing 2 and 8 represents the multiplicative structure in the problem, drawing 4 and 7 do not. Still most students accepted these drawings as appropriate and helpful as well as those illustrating the structure.

Conclusion

When young children are presented this type of task, they respond according to their perception of the task, with respect both to the problem statement and to the situation within which the problem solving takes place. The responses reported here indicate that young children accept a wide variety of visual representations of multiplicative problem statements. This variety is in accordance with the findings of Thomas, Mulligan and Goldin (1994).

When the children considered a drawing, the most important feature seemed to be that the number of objects in the drawing was the same as the result of the calculation. It was of less importance that the arrangement of the objects showed the underlying mathematical structure in the problem. This supports Nunokawa's (1994) claim that students do not inherit directly the structure in mathematical problems no matter how obvious or "intuitive" it might seem to well-educated mathematicians. A structure is rather imposed on the problem by the student and this perceived structure is based on the mathematical challenge and on the student's previous knowledge. This means that a diagram, as a device for representing the mathematical structure in a problem situation, may not function as such if the problem solver does not have the appropriate conceptual knowledge. It seems like the understanding of and the use of diagrams is deeply connected with other aspects of the student's conceptual knowledge. Teaching of diagrams must therefore be considered connected to the students' conceptual development, and it is likely to follow the same principles of teaching and learning as other parts of the mathematical concepts.

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A Discussion of Different Approaches to Arithmetic Teaching

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Arithmetic is central in mathematics teaching in the elementary years of schooling, not only for the knowledge of number that has applications in the 'real' world, but also because computation introduces increasingly complex mathematical processes. The operations of arithmetic provide opportunities for making connections within a growing knowledge of numbers, and for developing mental strategies through written symbolism. International and cross cultural studies have identified differences in achievements in arithmetic tests and have highlighted differences in teaching approaches that bring into question the appropriate emphasis to be placed on fundamental processes like counting, using place value and recording within standard algorithms. This paper will use the operation of division, including procedures for 'long' division, to exemplify fundamental differences in teaching approaches. In particular, it will take as examples the 'culturally cognate' European neighbouring countries of England and Holland.

In "A cross cultural investigation into development of place value concepts of children in Taiwan and the United States", Yang and Cobb (1995) reported that Chinese students' arithmetical conceptions were generally more advanced than those of American students even though place value was not explicitly taught as a separate topic. The notion that numbers are composed of units of ten and one gradually emerged as an established mathematical truth in the Chinese classroom, in contrast with the procedurally based manipulative experiences in the American classroom. The authors suggest that 'it might be fruitful to explore instructional approaches in which the construction of increasingly sophisticated place value concepts is treated as an extended problem-solving process rather than the acquisition of predetermined facts about number'. It is not only the direct teaching of place value that needs to be questioned as a basis for calculating with numbers, but the role of counting and the use of known and derived number facts in the development from mental to written strategies as problems become increasingly complex.

Different teaching approaches

In some countries the explicit understanding of place value is seen as crucial to the development of number knowledge. Vertical algorithmic recording may be introduced and structured materials used to illustrate procedures that enable two digit and larger numbers to be 'disaggregated' into tens and units (SCAA, 1997) so that operations may be related to single digit number facts. In Britain, for example, there is an early emphasis on written approaches that promote the use of place value concepts. 'Understanding of place value is central to pupils' learning of number....Progression in understanding about place value is required as a sound basis for efficient and correct mental and written calculation' (SCAA, 1997)

Other countries place emphasis on development of a more holistic characterisation of numbers and focus on mastery of counting to develop informal mental strategies. The Dutch 'Realistic' approach to arithmetic teaching, for example, reflects the work of Freudental (1973) who advocated linking up early maths activities to children's own informal counting and structuring strategies. Discovery of simple patterns and easy structures like counting in 2s, 5s and 10s is conceived as an important emergent mathematising activity. In a move away from structured materials the 100 number square and more recently the 'empty' number line (Beishuizen, 1993) have been used to work explicitly at the development of mental counting strategies in a process of 'progressive mathematisation'.

These differences between the British and Dutch approaches to arithmetic are clearly exemplified in the way the operation of division is introduced and taught. It is traditional in Britain for pupils to be introduced to the 'bus shelter' notation for division $4\overline{)36}$ at a stage when the problem may be solved by recall. Encouragement is given in text books to use this written algorithmic format, as illustrated by the first example in Nuffield 4 Teacher's Handbook (1992) which sets out the problems in vertical format $4\overline{)36}$, and places emphasis on the arrangement where the answer "9 is placed above the 6".

This recording will facilitate solution of problems with larger numbers like 639÷3 where a 'digitwise' approach provides an algorithm requiring limited understanding of the numbers involved. Indeed, this written algorithmic approach may later constitute a mental strategy based on the same 'disaggregation' of the dividend into digits.

In contrast, pupils in Holland are expected to work mentally only in the early stages of learning division and they do not meet a written approach until grade 4 (age 9-10) where very large numbers, e.g. 1670÷14, are used to make mental calculation without some written recording inappropriate. A popular textbook series 'Rekenen en Wiskunde' (Van Galen et al., 1988) developing the 'Realistic Mathematics Education' approach (Streefland, 1991), shows the clear distinction made between mental and written strategies. In the pupils texts, all division problems involve a single digit divisor are presented in context 'word problems' or in symbols with no numbers going beyond the facts of the times tables. Introduction of a written algorithm is deliberately postponed until pupils have a secure understanding of mental strategies for division.

> *"In grades 1, 2 and 3 there is no room for the standard algorithm. Mental arithmetic must be developed first, according to the realistic idea. If the algorithms are introduced in grades 2 and 3 mental arithmetic does not stand a chance, certainly not for weaker pupils, and arithmetic threatens to deteriorate to blind manipulation with numerical symbols - this at the expense of both pure arithmetic as well as the ability to apply it."*
>
> *(Treffers, 1991 p48)*

The teachers' guides in this Dutch series introduce problems with larger numbers through context problems like calculating the number of buses needed to transport 1128 soldiers when each bus has 36 seats (Streefland, 1991). Through discussion of such problems pupils are led through many different stages which relate their mental strategies with written recording of this thinking. Pupils are expected to make progress at different rates towards more 'abbreviated arithmetic methods' and greater efficiency in recording.

Organisational characteristics

In these two approaches there are contrasts in classroom organisation, timing for introducing written recording and motivation when introducing a written approach. British pupils practice written procedures using a standard algorithmic approach, as individuals or in small groups, early in their experience of division. Small numbers are initially used to develop a 'place value' recording system that has direct application to problems with larger numbers. Dutch pupils work together as a whole class and initially develop mental strategies and recall for division of small numbers. Only later is whole class discussion used to establish the way strategies may be recorded for larger numbers.

In understanding the contrasting teaching approaches in Britain and Holland these organisational characteristics including the use of group work and whole class teaching become relevant. It is common for pupils in Britain to work from an early stage independently with pencil and paper. To accommodate the needs of individuals who develop arithmetical understanding at different rates division may be introduced to individuals or small groups at different times throughout years 4, 5 and 6 (ages 8 - 11). This is in contrast to the tradition in Holland (and many continental countries) where whole class teaching involves all pupils starting division at the same time and early emphasis on mental strategies.

There has been much publicity recently in Britain encouraging whole class teaching but it must not be forgotten that such an approach is facilitated in many European countries by a policy of pupils at the lowest end of the attainment range repeating one or more years and by the withdrawal of pupils with special educational needs. Data presented by Prais (1997) contrasts English classes with 100% of pupils of the same chronological ages in the same class with only 58% chronologically 'correct' in corresponding classes in Switzerland where delayed entry into school and repeated years are common.

Partitioning down and Counting up

The fundamental difference in the two approaches referred to above is not only the stage at which written recording is introduced but also the way large numbers are dealt with and related to the facts already established for small numbers.

On the one hand, numbers are split/disaggregated into tens and units or HTU and practical activities are linked to this model of decomposition. Within such teaching, written algorithms are related to the use of structured apparatus like Dienes blocks (SCAA, 1997) and usually linked to a vertical system of written recording.

On the other hand, counting is used to build an understanding of patterns which help to characterise numbers as wholes in relation to each other. Support for this holistic approach to numbers has been identified in use of the 'empty' number line (Beishuizen, 1993) where 'steps' may be taken in different sizes and focus is placed on behaviour around the tens and hundreds boundaries.

Whether 'partitioning down' from the dividend or 'counting up' from the divisor, the division of large numbers can only become efficient if pupils recognise that numbers are made from constituent parts in different ways. In order to divide 639 by 3, for example, the digits 6, 3 and 9 corresponding to 600, 30 and 9 may be divided in turn by 3 with a standard written system 3)639 facilitating this solution strategy. Alternatively, 639 may be 'built up' in stages of 300 and 300 to give 600 (200 lots of 3) and then 30 (10 lots of 3) and 9 (3 lots of 3). Clearly there is a correspondence between these methods but strategically the approach is different. In either case it is important that pupils' overall number sense continues to be developed through connections with existing number knowledge and mental strategies and that efficiency in written formats is not developed at the expense of understanding.

A study in Britain

In a small scale study of 51 pupils in years 5 and 6 (9 to 11 year olds) in Britain, each pupil was asked to solve six division problems which were chosen to involve different types of solution strategy. Pairs of pupils were videotaped solving the division problems which were presented one at a time in symbols (e.g. 96 ÷ 4) on individual cards. After they had completed each problem, using pencil and paper for working, each child was asked to explain their solution procedures to the interviewer (and the camera). The same children were video taped again four months later attempting similar problems after they had received further instruction for division. This instruction included practice with the standard written algorithm. As expected, overall performance improved but this was not the case for all the problem types or for all the individuals tested.

Problem	% Successful (Oct/Nov)	% Successful (Feb/March)
1) 96÷4 / 96÷6	53%	47%
2) 34÷7 / 41÷7	11%	33%
3) 6000÷6 / 8000÷8	61%	87%
4) 4÷½ / 3÷½	35%	54%
5) 6÷12 / 4÷8	19%	16%
6) 68÷17 / 76÷19	25%	35%

Table showing results in tests (n=51)

Improvement from an unsuccessful attempt at 96÷4 to a successful attempt at 96÷6 was seen in 6 children (12%) and in 4 cases this involved a change from an informal counting strategy to use of the written algorithm. In a further 9 cases (18%), however, success with 96÷4 changed to failure with 96÷6 despite further instruction. In 2 cases the written algorithm was used successfully for 96÷4 but not for 96÷6. In the other 7 cases counting strategies involving multiples of 4 (or 6) were used for both problems and deterioration may be attributable to the change in divisor from 4 to 6. Further practice with multiplication facts may explain the improvement in the second type of problem (34÷7/41÷7) while improvement on the third type (6000÷6 / 8000÷8) showed evidence of more successful use of the written algorithm. Additional experiences with fractions may explain the better performance for the fourth problem type.

In the fifth problem type (6÷12/ 4÷8) the obvious error of interpreting the problem with numbers reversed (i.e. 12 ÷6/ 8÷4) was more prevalent in the second testing than in the first.

The final example was most interesting because out of the 13 pupils who were successful with the problem 68÷17 in the first test, more than half (7 individuals) switched to a written algorithmic approach that was not successful for 76÷19. Improvement overall was due to 12 individuals who were not successful with 68÷17 but successfully used a counting strategy involving adding 19s to solve 76÷19.

	improved	deteriorated	overall change
96÷4 / 96÷6	12%	18%	down 6%
68÷17 / 76÷19	14%	24%	up 10%

Table showing changes in performance between first and second test

These two examples give cause for concern because of the large deterioration in performance compared with the improvement after instruction.

The overall performance for the individual children across the six items showed improvement in 24 cases (47%) but for 10 individuals (20%) they achieved fewer correct answers in the second test than in the first. In one notable case, David was correct for 4 items in the first test using informal strategies but used the written algorithm for all six items in the second test and was successful only with 8000÷8. Although this is a small sample and the items were not identical across both tests there are some aspects from the results that cause concern, particularly where a successful informal strategy is replaced by an unsuccessful application of a written algorithm.

Formalising procedures

In a comparative study of mental computation performance in Australia, Japan and the United States, Mc Intosh, Nohda, Reys and Reys (1995) concluded that 'early concentration on formal computation for all children, whether mental or written, may not be beneficial in the longer term. Indeed, it may be time wasted which could profitably have been used on material at a more appropriate conceptual level".

Formalising strategies for arithmetic problem solving is at the heart of arithmetic teaching and a fundamental question appears to be the relevance of place value as a central organising characterisation for work with large numbers. Structured materials have been very influential in both the U.S. and the U.K. (Stern & Stern, 1971; Thompson, 1997) to support place-value approaches with corresponding written (vertical) procedures. Stern and Stern emphasized that this method should be preferred because "the important principles of mathematics must be demonstrated" (p. 223).Counting is seen as a mechanical and meaningless activity, which should not be stimulated but instead should be replaced (as soon as possible by arithmetic blocks and following the formal HTU structure). Beishuizen and Anghileri (1998) argue to the contrary that counting needs to be developed and extended to include abbreviated forms of counting in units other than ones. The empty number line provides a model that is used in Holland to teach mental strategies for addition and subtraction enabling imaging of moves forward or backward in steps of tens and hundreds as well as convenient units that may relate to any particular problem. This develops pupils' familiarity with numbers and their construction in a variety of ways from different component parts. It has also been argued that the calculator can be an important support for developing understanding of the ways different numbers are constituted. Ruthven (1998) identifies pupils confidence that is particularly related to the Calculator Aware Number (CAN) curriculum in which explicitly taught mental methods were based on familiarity with 'smashing up' numbers or 'breaking down' numbers i.e. disaggregating them into convenient components. Anghileri (1998) refers to this process as 'chunking' and notes the way 'chunks' will vary across different problems. For example 96 ÷ 4 will be facilitated by 'chunking' 96 into 80 and 16 while 96 ÷ 6 will be better related to 96 as 60 and 36.

The role of recording

In the two approaches contrasted in this report, through place value or extended counting, written recording appears to be an important consideration. Ruthven (1998) identifies two distinct purposes for written recording: "to augment working memory by *recording* key items of information" and "to cue sequences of actions through *schematising* such information within a standard spatial configuration". In the case of long division it appears that recording methods can 'direct and organise' the solution strategy or 'follow and record' steps in a mental strategy. Of course these distinctions are not clear cut but teaching of long division appears to illustrate well two diverse approaches with the same objective of curtailing procedures and ultimately producing an efficient recording process that will have application to any problems. With growing interest in international comparisons of performance in arithmetic more collaboration is needed in analysing and sharing the principles underlying teaching approaches and more research is needed to evaluate the effectiveness of such different approaches.

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A MODEL FOR ANALYSING THE TRANSITION TO FORMAL PROOFS IN GEOMETRY

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Summary. *This report sketches a model for interpreting the processes of exploring geometric situations, formulating conjectures and possibly proving them. It underlines an essential continuity of thought which rules the successful transition from the conjecturing phase to the proving one, through exploration and suitable heuristics. The essential points are the different type of control of the subject with respect to the situation, namely ascending vs. descending and the switching from one to the other. Its main didactic consequence consists in the change that the control provokes on the relationships among geometrical objects. The report frames the research in the existing literature (§1) and exposes the main points of the model through the analysis of a paradigmatic case (§§ 2,3); in the end (§4) some partial conclusions are drawn.*

1. Introduction.

In the current literature on mathematics education, the concept of proof is examined in a wide sense, which goes beyond the narrow formal one; in fact, also explorations, conjectures, argumentations produced by novices and experts while solving problems, as well as *semi-rigorous, zero-knowledge, holographic proofs* (see Hanna, 1996), are taken into account, because of their interest in the pragmatic of proof (Hanna & Jahnke, 1993). However, in real school life, even if proof is generally considered central in the whole mathematics, it does not enter all the curriculum, but it is restricted almost exclusively to geometry (Hanna, 1996). The processes through which pupils and experts approach proofs are analysed from different points of view and by means of different tools. First, at least two components are considered crucial for focusing the meaning of proof, namely a ***cognitive*** and a ***historic-epistemological*** one (Barbin, 1988; Balacheff 1988, 1991; Simon, 1996; Harel, 1996; Mariotti et al., 1997). Of course the two components can be separated only for reasons of theoretical analysis; on the contrary, they are deeply intertwined in the reality (Hanna, 1996) and both must be considered in order to tackle suitably the didactic of proof (even if different authors underline more one or the other of them). Second, the production processes of proofs are analysed pointing out ***continuity*** and ***discontinuity*** features both from the epistemological and the cognitive point of view; the question is particularly intriguing when one considers the relationships between the argumentative, informal side and the discursive, formal one of a proof (in the wide sense of the word). For example, the *transformational reasoning* of Simon, 1996, the *cognitive unity* of Mariotti et al., 1997, all underline a substantial continuity from a cognitive point of view. The issue of continuity from an epistemological point of view has been faced in Polya, 1957, Barbin, 1988 and Thurston, 1994. Moreover, some authors, like Duval, 1991, mark the dramatic epistemological and cognitive gap between argumentation and proof: Duval tackles it from a didactic point of view using suitable semiotic mediators, namely graphs for representing the formal deductions. In an intermediate position we find Harel, 1996 with his *students' proof schemes* and Balacheff, 1988, who, following the analysis of Lakatos, stresses the big epistemological discontinuities, which can be overcome by pupils, insofar as they become able to pass from the *naive-empiricist* way of looking at mathematical sentences towards a more *formal* approach, through the discovering of the so called *generic example*. The explicit or implicit attitude of the teacher towards the question 'continuities vs. discontinuities', both from a cognitive and an epistemological point of view, reveals to be crucial for planning the didactic of proof in the class (for examples of concrete approaches see Balacheff, 1988; Duval, 1991; Mariotti et al., 1997). The problem becomes even more intriguing when new technologies are taken into account and such softwares as Geometer's Sketchpad, Cabri-Géomètre, Derive, Excel or others are used in the class as tools for exploring situations, making conjectures and validating the same process of proving theorems.

Within this research issue, a crucial point consists in analysing the delicate phase of transition to the formal side, exploiting its connections with the informal one. Important variables for such an analysis are: the mathematical area, for ex. geometry, algebra, analysis, etc.; the modalities after which the problem is given, namely exploring an open situation vs. proving a given statement; the environment, namely paper and pencil vs. computer (for ex. a Cabri setting). Our research group has been studying the above problem for two years in the area of elementary geometry, making experiments in different environments with high-school and college students, as well as with their teachers. In this report we expose a theoretical model we elaborated to investigate the transition to the formal side. It is based on a careful analysis of processes of thought in experts or clever students who explore open problems in paper and pencil environment. It is used as a key to interpret processes of thought in pupils of different levels who solve geometrical problems in different environments and with different modalities. The main sources for this model are the papers quoted above, which analyse the relationships between conjecturing and proving under the issue of continuity. In particular, we are indebted to Gallo, 1994, for the notion of *ascending/descending control* and to Mariotti et al., 1997, for that of *dynamic exploration*, which supports the *selection/specification of conjectures* in the form of conditionals and rules the passage to the proof construction, by implementing the logical connections of sentences. We illustrate the model by means of a paradigmatic example, which is exposed in §2 and commented in §3.

2. A paradigmatic example.

We expose the protocol of solution given by a teacher to the following problem:

Problem. *Given a quadrilateral ABCD and a point* P_0, *construct the point* P_1, *symmetric of* P_0 *with respect to A,* P_2 *symmetric of* P_1 *with respect to B,* P_3 *symmetric of* P_2 *with respect to C,* P_4 *symmetric of* P_3 *with respect to D. Determine which conditions the quadrilateral ABCD must satisfy so that* P_0 *and* P_4 *coincide.*

The subject solving the problem used pencil, paper and (sometimes) ruler; he was invited to use only elementary mathematics and to think aloud: an observer took notes of his comments (which are written in *italics*, while observer's comments are in bracket parenthesis). The solution process has been divided into 15 episodes, which lasted about six minutes in the whole; a minor episode (n.6) has been skipped, because it is a detour not relevant for our analysis; references are to figures at the end of the protocol. The comments on the protocol are given in §3. (S = subject) .

1. [S draws very rapidly and sketchily, without using the ruler: fig.1].

2. *"I'll check for a simpler case, with only three points"* [S sketches a figure with triangles instead of quadrilaterals, i.e. D and P_4 disappear] *"I do not see anything".*

3.*"I consider a particular case, which is easier: the rectangle"* [S sketches fig.2]. *"Perhaps it closes in the rectangle's case"* [in the figure it is dubious if P_0 and P_4 coincide (P_0, P_1, P_2, P_3, P_4 close) or not, because the figure has been drawn by hand].

4.*"I can't see that in this way. I redraw a very different case, always with rectangle: P_0 far away from A"* [S draws fig.3 without ruler but with more attention, with a smaller rectangle but with P_0 far from A: P_0 and P_4 seem to coincide].

5.*"I see the Varignon's case in the opposite way"* [Varignon's theorem is a classic Cabri problem, well known to S; it says that, given a quadrilateral, if K, L, M, N in order are the middle points of its sides, then the KLMN is a parallelogram; successively asked, S says that he meant that he saw the Varignon configuration, with K, L, M, N as the rectangle of fig.3] .

[S looks carefully at the figure] *"However I realise it's not so".*

6. [In this short episode S tries to follow another idea, but he abandons it soon].

7. *"Let me draw it better"* [S draws fig.4 with the ruler and with great care]. *"I see Varignon's case applied to crossed quadrilaterals, 'cause I've drawn all segments completely"* [S drew full segments between P_0, P_1, P_2, P_3, P_4] .

8. *"Now I am going to use the analytic method. I imagine the problem has already been solved. In my mind I anticipate that it's Varignon"* [By analytic method S means the method of *Analysis* due to Pappus (see Panza, 1996); S redraws a figure like fig.4, using the ruler; but now he first draws points P_0, P_1, P_2, P_3, then A, B, C, D as midpoints of the sides P_0P_1, P_1P_2, P_2P_3, P_3P_0; in all previous drawings S drew A, B, C, D first and then P_0, P_1, P_2, P_3, P_4].

9. *"I see it's a rectangle again"* [In fact, in fig.4, even if S started from 'generic' P_0, P_1, P_2, P_3, P_4, the quadrilateral ABCD looks like a rectangle]. *"I conjecture that if it is a rectangle it will close".*

10. *"I'll prove it. I'm guided by Varignon's proof. It results that AB // CD // P_0P_2 and BC // AD // P_1P_3. Now I look at the figure again to prove it's a rectangle."* [He looks at the figure... he draws AC, BD...].

11. *"...hem...I reconsider AB // P_0P_2... and I observe that [the angle] ABC is equal to [the angle formed by the lines] P_0P_2, P_1P_3. I come to believe that in general it isn't a rectangle: I look for a counterexample. I start from P_0, P_1, P_2, P_3 and draw ABCD carefully".* [see fig.5].

12. *"It's a parallelogram. The proof is done! I write it down".*

13. *"I know that given a quadrilateral ABCD (even crossed), the quadrilateral constructed on the midpoints is a parallelogram, 'cause of Varignon. Now let us consider a parallelogram ABCD. If P_0,P_1,P_2,P_3,P_4 are built as symmetric then the*

thesis is that the points P_0 and P_4 coincide. I go back to the figure to prove it" [S writes the key words of his theorem arranged as hypothesis and thesis on another sheet of paper; then he comes back to the sheet with his drawing...].

14. *"It's a synthesis!"* [...S takes a new sheet of paper and draws a figure that is deliberately 'wrong', by hand without a ruler, see fig.6].

"I consider P_0, P_1, P_2, P_3 with the resulting quadrilateral: I construct the first three midpoints which are A, B, C. [While speaking, he draws fig.7] *Then I construct D', midpoint of P_0P_3. ABCD' is a parallelogram because of Varignon".*

15. [Now S draws fig.8 and writes down what he is saying] *"ABCD is a parallelogram too, by hypothesis. If $D \neq D'$ then $CD \neq CD'$, but they are both parallel to AB. It's absurd! Then ABCD and ABCD' are congruent. Therefore P_0 and P_4 coincide".*

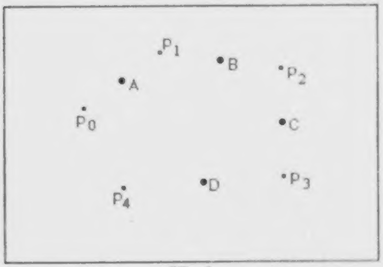

Fig.1

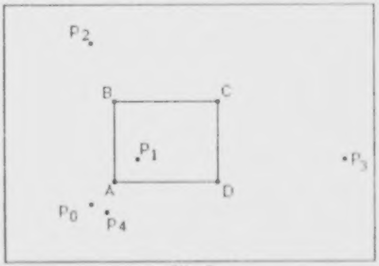

Fig.2

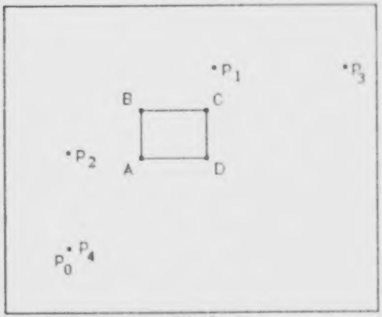

Fig.3

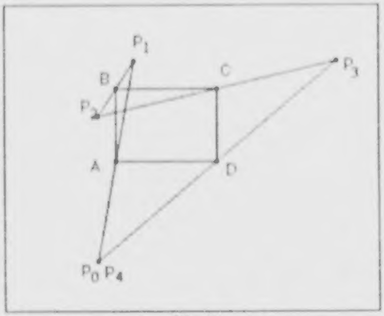

Fig.4

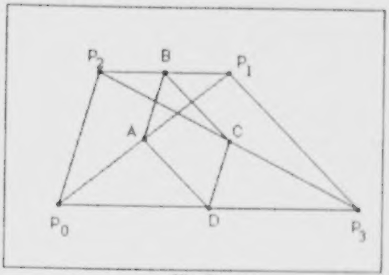

Fig.5

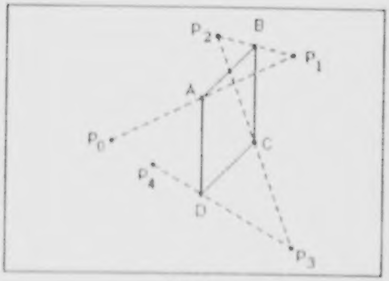

Fig.6

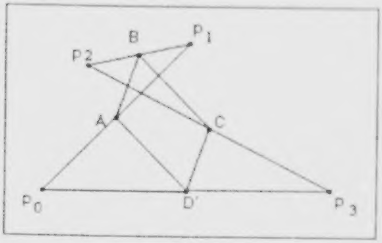

Fig.7

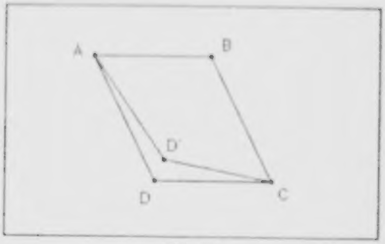

Fig.8

3. The theoretical model.

It is now time to explicit our model of transition from conjecturing to proving (see the protocol as a paradigmatic example).

As a first working hypothesis, which we shall modify during the exposition, we use that of Mariotti et al., 1997 (but the responsibility of the interpretation is only due to this report's authors). High and middle level subjects, who explore geometrical problems in different environments in order to conjecture and to prove theorems (within their own theoretic framework) show successively two main modalities of acting, namely: *exploring/selecting a conjecture* and *concatenating sentences logically*. In fact, any process of exploration-conjecturing-proving is featured by a complex switching from one modality to the other and back, which requires a high flexibility in tuning to the right one. Our aim is to analyse carefully how the transition from one modality to the other does happen, using the protocol above: its dynamic has been divided into four main phases, each corresponding to a different modality or transition. At the end, the picture of the transition will appear and we shall rephrase the working hypothesis in a suitable way (§4).

PHASE 1. Episodes 1-3 show a typical exploring modality, with the use of some heuristics to guess what happens working on a particular example (ep.3), hence selecting a conjecture. The conjecture in reality is a working hypothesis to be checked: its form is far from a conditional statement and to confirm it new explorations are made by using a new heuristic principle (namely: *choose very* *different data, to check the validity of the conjecture*, ep.4). The phase culminates in ep.5: some of its general features are described in Mariotti et al., 1997, specifically the *internalisation of the visual field* (the subject 'sees'), and the *detachment* from the exploration process (which is seen from the outside); the situation is described by the subject in a language which has a logic flavour (ep.5), but it is not phrased in a conditional form (if...then) nor it is crystallised in a logical form: in fact, the subject expresses his hypothesis (which is more stable and sure than that of ep.3) not yet as a deductive sentence, but as an *abduction*, namely a sort of *reverse deduction*, albeit very different from an induction (Peirce, 1960) (1). In fact the subject *sees* (with his mind's eyes, because of the internalisation of his visual field) *what rule it is the case of*, to use Peirce language. Namely, he selects the piece of his knowledge he believes to be right; the conditional form is virtually present: its ingredients are all alive, but their relationships are still reversed, with respect to the conditional form: the direction after which the subject *sees* the things explored in the previous episodes is still in the stream of the exploration: the control of the meaning is *ascending* (we use this term as in Saada-Robert, 1989 and Gallo, 1994). It is in the stream of the preceding exploration that the negative validation at the end of ep.5 happens. Ep.7 is still in the same stream of thought; now the heuristic is: *draw better to see better*; indeed it is the last drawing (fig.4) which allows the second abduction (ep.7): it is interesting to observe that the *hypothesis* changes (now the quadrilateral is crossing) but it is still in the reversed abductive form.

PHASE 2. Ep.8 marks the switching from the abductive modality to the deductive one: the meta-comments in the protocol show this clearly; but this change is showed also by the way in which the figure is drawn: see the observer's comments. Now the control is *descending* and we have an exploration of the situation, where things are looked at in the opposite way. Ep.9 shows this: exploration now produces as output the figure which in previous explorations was taken as input. The reversed way of looking at figures leads the subject to formulate the conjecture in the conditional form. Now the modality is typically that of a logical concatenation.

PHASE 3. Now in the new modality suitable heuristics can be used, namely *look for similar proofs* (ep.10): this task seems straightforward for the subject and so does not generate any further exploration, at least as far as parallelism of sides is concerned. Some exploration (with descending control) starts at the end of ep.10, for proving that it is a rectangle, but it does not work, so a new exploration, after a new selection (concerning angles) starts with ep.11. Here the descending control is crucial: it allows the *detached* subject to interpret in the 'right' way what is happening: it is not an abduction (what rule it is -possibly- the case of) but a counterexample (what rule it is not -surely- the case of); the switched modality has started a new exploration process, which culminates in the final conjecture of ep.12. PHASE 4. It is the real implementation of logical connections in a more global and articulated way than the local concatenation of statements, which featured the previous conjecturing phase. Here detachment means to be a true *rational agent* (Balacheff, 1982), who controls the products of the whole exploring and conjecturing process from a higher level, selects from this point of view those statements which are meaningful for the very process of proving and rules possible new explorations. In this last phase, conjectures are possibly reformulated in order to combine better *logical concatenations* (ep.13) and new explorations are made to test the latter: looking at what happens word by word, this exploration is not very far from those made under ascending control. It is the sense attached to them by the rational agent to change deeply the meaning of what happens. A typical example is ep.14, where a 'wrong' figure is drawn (fig.6) in order to explore the situation, anticipating that it is an impossible case: during the episode a second figure is drawn (fig.7), where the 'logical impossibility' has changed the relationships among the points: in fact the old point D has been substituted by a new point D', which incorporates in a positive way the logical impossibility. The control is typically descending and global; in fact a proof by contradiction is tackled: the sense of the logical relationships among the drawn objects produces a 'new' situation, which is explored. In ep.15, the 'old' and the 'new' situations are put together by the rational agent (fig.8), who can draw the conclusion by contradiction.

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(1) The example given by Peirce is illuminating. (Peirce, 1960, p.372). Suppose I know that a certain bag is plenty of white beans. Consider the following sentences: A) these beans are white; B) the beans of that bag are white; C) these beans are from that bag. A deduction is a concatenation of the form: B and C, hence A; an abduction is: A and B, hence C (Peirce called hypothesis the abduction). An induction would be: A and C, hence B.

4. Some partial conclusions and related problems.

Our model is somehow different from the starting working hypothesis: in fact the exploration and selection modality is a constant in the whole conjecturing and proving processes; what changes is the different attitude of the subject towards her/his explorations and the consequent type of control with respect what is happening in the given setting. *It is the different control to change the relationships among the geometrical objects, both in the way they are 'drawn' and in the way they are 'seen'.* This seems essential for producing meaningful arguments and proofs. Also detachment changes with respect to control: there are two types of detachment. The first one is very local and marks the switching from ascending to descending control through the production of conjectures formulated as conditional statements (that is *local logical concatenations*) because of some abduction, like in ep. 5 and 7. The second one is more global and we used the metaphor of the *rational agent* to describe it: in fact it is embedded in a fully descending control, produces new (local) explorations and possibly proofs (that is *global logical combinations*), like in ep. 10, 11, 12, 14. *The transition from the ascending to the descending control is promoted by abduction*, which puts on the table all the ingredients of the conditional statements: it is the detachment of the subject to reverse the stream of thought from the abductive to the deductive (i.e. conditional) form, but this can happen because an abduction has been produced. *The consequences of this transition are a deductive modality and the new relationships among the geometrical objects of the figures*, as pointed out above (ep. 8). The inverse transition from descending to ascending control is more 'natural': in fact as soon as a new exploration starts again (ep. 14), control may change and become again ascending, even if at a more local level (with the rational agent who still control the global situation in a descending way). In short, the model points out an essential continuity of thought which rules the successful transition from the conjecturing phase to the proving one, through exploration and suitable heuristics, ruled by the ascending/descending control stream. The most delicate cognitive point is the process of abduction, crucial for switching the modality of control; the most relevant didactic aspect is the change in the mutual relationships among geometrical objects, which are the essential product of such a switching. Many scholars, with a different language, exploited carefully various aspects of the way in which the switching can be realised by pupils in the class. In another ongoing research, we use our model to study how the Cabri environment can support pupils in getting the above switching and changing of the relationships among the geometric objects.

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DRAGGING IN CABRI AND MODALITIES OF TRANSITION FROM CONJECTURES TO PROOFS IN GEOMETRY

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Summary. *In this report we analyse some modalities that feature the delicate transition from exploring to conjecturing and proving in Cabri: we use a theoretical model that works in other environments too. We find that the different modalities of dragging are crucial for determining a productive shift to a more 'formal' approach. We classify such modalities and use them to describe processes of solution in Cabri setting, comparing it with the pencil and paper ones.*

1. Introduction.

The literature on computers as cognitive tools (Dörfler, 1993) which modify the learning of mathematics because of their specificity and 'situativeness' as learning environments (Hoyles & Noss, 1992) is especially rich for Cabri-géomètre (Laborde, 1993; Balacheff, 1993; Hölzl, 1995, 1996). In particular, several researches which analyse specific components of Cabri's *epistemological domain of validity* (Balacheff & Sutherland, 1994) point out that for learning geometry in Cabri environments the dialectic figures vs. concepts (Mariotti, 1995 and Laborde, 1993) and perceptual activity vs. mathematical knowledge (Laborde & Strässer, 1990) is essential. Typically, a geometrical problem cannot be solved only remaining at the perceptual level of figures on the screen, even if their graphical space is provided with movement as a further component (Laborde, 1993): a conceptual control is needed, and it requires some pieces of explicit knowledge. Dragging, which has a complex feedback with the visual perception and the movements of the mouse, is a crucial instrument of mediation between the two levels (Hölzl, 1995): its function consists in validating procedures and constructions built up using the menu commands (Laborde & Strässer, 1990, p.174; Mariotti et al., 1997). While dragging, pupils who make constructions or explore geometric situations often switch back and forth from figures to concepts and an evolution of their attitudes from the *empirical* to the *theoretical* level can possibly be generated in the long run (Balacheff, 1988; Mariotti et al., 1997; Laborde, 1997). This switching (and the generated evolution) can also be observed in pencil and paper environments, particularly in experts' and clever students' performances; it is crucial in all environments insofar as it makes possible for pupils to manage the big gap between the status of knowledge based on drawings and the one which refers to geometrical concepts, sustaining them in the solution process and avoiding stumbling-blocks. Our research group developed a model for analysing such processes of switching in pupils who explore geometric situations, who produce conjectures and prove them. To get it, in the last two years we carried on teaching experiments in different environments with high-school pupils, college students, some of their teachers and we collected many empirical data. In this report we use our model for describing the switching modalities in pupils who use Cabri (1) to solve geometric problems and for contrasting them with the modalities of pencil and paper environments. Such a description will isolate in a transparent way some components of Cabri's *epistemological domain of validity*, which become important didactic variables for our project of teaching geometry at high-school level with a multi-medial approach (pencil and paper, Cabri, geometrical machines, etc.). The major results of our research are two. First, dragging behaviours change according to the specific epistemological and cognitive modalities after which pupils develop their control (and consequently make their actions) in Cabri; hence, looking at dragging modalities can give an insight into other inner and more theoretical variables. Second, in some pupils, particularly in those who produce good conjectures while exploring open situations, the modality of dragging involves different specific features, such as the so called *lieu muet* (dummy locus), described below (§2).

2. Explorations and constructions in Cabri.

Before discussing some concrete examples, we sketch very shortly the main points of our model. We consider tasks of exploring *open* geometric problems (Arsac, 1988) in order to select/formulate conjectures and possibly to prove them. The model points out an essential continuity of thought, which features the successful transition from the conjecturing phase to the proving one, through exploration and suitable heuristics, ruled by what we call an *ascending/descending control stream* (see Saada-Robert, 1989 and Gallo, 1994). The process of switching from one control modality to the other is a delicate cognitive point, which has also a relevant didactic aspect: in fact it is deeply intermingled with the change in the mutual relationships after which the geometrical objects of the situation are seen. It is precisely in these two aspects that one can observe different dynamics between 'pencil & paper' and 'Cabri' environments. In both, the transition is ruled by *abduction*, which will be explained below; but while in the former the abductions are produced because of the ingenuity of the subjects, in Cabri the dragging process can mediate them: our model allows to describe how Cabri can support pupils in getting the above transition.

We distinguish between 'constructions' and 'open problems' explorations', which correspond to two different modalities of using Cabri. The former consists in drawing figures through the available commands of the menu, because of a construction task, which is considered solved if the figure on the screen passes the *dragging test*: the Cabri figure will not be messed up by dragging it (it has been studied by Mariotti et al., 1997). For the latter, let us illustrate it with an example. Consider the following problem to be solved in Cabri:

> *Let ABCD be a quadrangle. Consider the bisectors of its internal angles and their intersection points H, K, L, M of pairwise consecutive bisectors. Drag ABCD, considering all its different configurations: what happens to the quadrangle HKLM? What kind of figure does it become?*

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(1) All our experiences refer to Cabri I, MS-DOS version.

This is a typical example of the open problems we use in our experiment, which is being carried out in the second year of a 'Liceo Scientifico' (pupils aged 15), aimed at teaching geometry with a multi-medial approach (in this specific case, with Cabri). Our example will illustrate other modalities of dragging, namely: (i) *wandering dragging*, that is dragging (more or less) randomly to find some regularity or interesting configurations; (ii) *lieu muet dragging*, that means a certain locus *C* is built up empirically by dragging a (dragable) point P, in a way which preserves some regularity of certain figures.

We analyse the data collected from a class of 27 students, who have already learned some Euclidean geometry the year before; the exposed activity takes place in a two-hour lesson. One hour is devoted to the work with Cabri (two students for each computer): having created a paper and pencil drawing of the geometrical situation, the pupils go on working in Cabri and making conjectures. The second hour is devoted to a mathematical discussion about the groups' discoveries: groups show their discoveries to the class, using a data-show, and the teacher orchestrates the discussion (according to the methodology illustrated in Bartolini Bussi, 1996) so that students can move towards more general statements. The analysis of the collected material shows three different ways of using Cabri in order to solve the problem, corresponding to the three dragging modalities mentioned above: *lieu muet, dragging test, wandering dragging*. A case in point of the first two is the protocol of Group-A (high-level students):

1. The pupils start to shape ABCD into standard figures, apparently following an implicit order 1-2 (i.e.: when ABCD is a parall., HKLM is a rectangle), 2-3, 3-4.

EXTERNAL FIGURE ABCD	INTERNAL FIGURE HKLM
Parallelogram (1)	Rectangle (2)
Rectangle (2)	Square (3)
Square (3)	The bisectors and the diagonals all pass through one point. (4)

2. As soon as they see that HKLM becomes a point when ABCD is a square, they consider it an interesting fact, therefore they drag ABCD (from a square) so that H, K, L, M keep on being coincident (*lieu muet* exploration).

3. They realise that this kind of configuration can be seen also with quadrilaterals that apparently have not got anything special; so they look for some common properties to all those figures which make HKLM one point. Paying attention to the measures of the sides of the figure ABCD (which appear automatically next to the sides and change in real time, while dragging along the *lieu muet*), they see that the sum of two opposite sides equals the sum of the other two; they remember that this property characterises the quadrilaterals that can be circumscribed to a circle.

4. Using the Cabri menu, they construct the perpendicular lines from the point of intersection of the angle bisectors to the sides of ABCD: they 'see' that this point has the same distance from each side of ABCD, then they draw the circle which has this length as radius: it is the circle inscribed in ABCD. They formulate the following 'conjecture': *If the external quadrilateral can be circumscribed to a circle, then its*

internal angle bisectors will all meet in one point, so the distances from this point are equal and the sum of the opposite sides is equal too.

5. They wonder whether this works even if they begin their construction with the circle: they construct a circle, a quadrilateral circumscribed to this circle, its angle bisectors and they observe that all of them meet in the same point; afterwards, they write down this 'conjecture': *If the internal angle bisectors of a quadrilateral all meet in the same point then the quadrilateral can be circumscribed to a circle.*

Let us examine carefully episodes 2 and 3. First, pupils look for an object more *generic* than a square (which is thought as "trivial" and "too easy a figure"), such that points H, K, L, M still coincide: they do that by *lieu muet dragging*. Then, their attitude changes: they look at the figure in Cabri without moving anything, try to discover some rule or invariant property under the *lieu muet* dragging, select 'which rule it is the case of' in their geometrical knowledge; this phase is marked by a continuous switching from figures to theory and back. Some general features of this new attitude are typical and described also in Mariotti et al., 1997: specifically we see in these pupils the *internalisation of the visual field* (the subjects 'see'), and the *detachment* from the exploration process (which is seen from the outside). Moreover, it is typical also that the subjects express their hypothesis not yet as a deductive sentence, but as an *abduction*, namely a sort of 'reverse deduction', albeit very different from an induction (Peirce, 1960) (2). In fact the subjects 'see' what 'rule' this is the case of, to use Peirce language. Namely, their visual field has been internalised in order to find a property which can help them to classify the figures into something they know; they select the part of their geometrical knowledge they judge as the right one. The conditional form is virtually present: its ingredients are all alive, but their relationships are still reversed, with respect to the conditional form; the direction after which the subjects 'see' things is still in the stream of the exploration through dragging, the control of the meaning is *ascending*, namely they are looking at what they have explored in the previous episodes in an abductive way. Control direction changes in ep.4: here students use the construction modality (and the consequent *dragging test*) to check the hypothesis of abduction and at the end they write down a sentence in which the way of looking at figures has been reversed. By *lieu muet* dragging, they have seen that when the intersection points are kept to coincide the quadrilateral is always circumscribed to a circle; now they formulate the 'conjecture' in a logical way, which reverses the stream of thought: 'if the quadrilateral is circumscribed then the intersection points coincide'. It is not a mistake! This episode marks the switching from the abductive to the deductive modality: now the control is *descending* and things are looked at in the opposite way. In ep. 5 the descending control continues; exploration now produces as output the figure which in previous dragging was taken as input: the pupils now construct a figure with the underlined property in order to validate the conjecture itself and check whether the figure on the screen passes the *dragging test*. So they come to explicit a conjecture expressing a sufficient and necessary condition "if... and only if..." in a conditional form, even if they are not able to summarise it into one statement only. Hence, at the end of their resolution process they have got all the elements they need to prove the statement.

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(2) The example given by Peirce is illuminating. Suppose I know that a certain bag is plenty of white beans. Consider the following sentences: A) these beans are white; B) the beans of that bag are white; C) these beans are from that bag. A deduction is a concatenation of the form: B and C, hence A; an abduction is: A and B, hence C (Peirce called hypothesis the abduction). An induction would be: A and C, hence B.

We can find some interesting elements also in the discussion which immediately followed the activity in Cabri (St 8, 9 = students of Group-A):

[...] St 9: "Well, we can find many other figures in which all the bisectors meet in the same point, in some quadrilaterals that apparently haven't got anything special. (1) [he moves the figure by *lieu muet* in order to have a generic quadrilateral in which H, K, L, M are coincident] *But, if we draw a circle ...no, first of all let draw a perpendicular line through one of these points* [H, K, L, M] *to one of the sides of ABCD* (2) [he draws the perpendicular from L to DC] *and consider the intersection point ...* [he draws], *we notice that this quadrilateral is circumscribed to a circle, then since it is a circle all the radius are equal and all the distances from the sides of ABCD are equal too..."*

St 8: "... all these centres are coincident ..." [...]

St 9: "If a quadrilateral can be circumscribed to a circle, all its angle bisectors meet in the same point."

St 8: "We proved the same thing but starting from a circle too (3); we drew the tangent lines and we came to the same conclusion." [...]

These students recollect what they have just found out reversing the exploration process: the descending control is ruling their thinking in the discussion phase. It is important to underline which concerning Cabri elements are still present in their words, which now are spoken from a detached point of view (numbers refer to the sentences in the discussion): (1) The *lieu muet dragging*, which allows them to move from a square to a more generic object that keeps H, K, L, M coincident [they are probably moving along a diagonal of the square]. (2) The construction activity (perpendicular line), with the *dragging test*, which supports their reasoning towards proof. (3) The "only if" form of their conjecture. Here we have a second form of detachment, fully embedded in the descending control stream, which we call the *rational agent* (Balacheff, 1982): they control the products of the whole exploring and conjecturing process from a higher level, selecting from this point of view those statements which are meaningful for the very process of proving and rule possible new explorations. They are reversing again the way of looking at the relationships among the objects: however this is not an abduction, but a logical concatenation of the 'only if' part (see Mariotti et al., 1997 as regards the 'only if' reasoning).

We also found another modality of dragging (*wandering dragging*), which we will illustrate sketching Group-B strategies. These students (of middle level) have a dynamic approach to the problem as well: they begin by dragging the vertices of ABCD at random and observing what happens to HKLM. As soon as they see something interesting about HKLM, such as a known or a 'strange' shape (for ex. a crossing quadrilateral), they stop moving. Then they go on by (*lieu muet*) dragging ABCD so that HKLM keep the same shape and they look at ABCD, trying to find out what kind of quadrilateral it is. We notice an evolution in their way of using the drag mode in Cabri: at first they seem to move the drawing just because Cabri allows them to do so, they haven't got any plan in their mind and move points at random; then they change their behaviour and move points in such a way as to keep a certain property of the figure, e.g. along a fixed direction. They continue switching from the first mode to the second one, every time they find an 'interesting' shape of HKLM. Hence the *lieu muet dragging* can be seen as a *wandering dragging* which has found its path, as some possible regularity has been discovered, at least at a perceptual level: both the dragging modalities are in the same stream of thought, namely in the ascending control one; at the opposite side we find *dragging test*, which is typical of descending control (albeit it can be used at different levels of sophistication).

3. Dragging by *lieu muet* as a reorganiser.

The protocols above are very important, because they clearly show how the dialectic between the different modalities of dragging can deeply change the relationships among the geometrical objects of the situation; so through the analysis of the dragging modalities used by pupils we can observe how such a shift takes place. In particular we shall concentrate on *lieu muet* modalities. A *lieu muet* can act both as a *logical reorganiser* (Pea, 1987 and Dörfler, 1993) and as a *producer of* new powerful *heuristics* (Hölzl, 1996). The former shows a new, intriguing way after which dragging can act as a mediator between figures and concepts (Hölzl, 1996), namely at a deeper and unexpected level of conceptual knowledge; the latter makes accessible some aspects of such a reorganised knowledge at a perceptual level and in a strongly 'situated' way, so it seems to support a 'situated abstraction' in the sense of Hoyles & Noss, 1992 (compare group-A protocol with the example in Hölzl, 1996).

Let us sketch the kind of logical reorganisation that the *lieu muet* encompasses: it shows a new and wide component of Cabri's *epistemological domain of validity*. The example of exploration showed in our protocols illustrates this in a paradigmatic way; a lot of explorations described in the literature seem to be coherent with our analysis: e. g. the cases discussed in Hölzl, 1996, where he observes a shifting of perspective in students "from the constructions of certain points to the interpretation of certain loci" (p.181).

The *lieu muet* 'shows' a new logical relationship between points and figures, which adds to the usual functional dependence of the kind variables-parameters, where some constructed objects depend in their construction on others which are considered as 'given'.

The new relationship consists in the fact that: (i) a certain locus C is empirically built up (see group-A protocol, as well as example at p. 176 in Hölzl, 1996) thanks to a feedback given by the preservation of some 'regularity' in drawn figures and the movement of the mouse dragging a (dragable) point P in a suitable way (which means precisely that P describes *C* as a *lieu muet*); (ii) when the point P runs on *C* some corresponding figures F(P) satisfy some regularity, invariance or rule (in the example, for each P belonging to the empirical curve *C*, the bisectors of the corresponding quadrilateral Q(P) meet in the same point). To use a mathematical language, the (usually algebraic) variety *C* (usually of dimension 1, that is a curve) parameterises (all or some of) the figures of the situations in a way which is perspicuous for the problem to solve. This parameterisation of course is given only through dragging and not by equations: if made explicit they would express the relationships found empirically by dragging in the language of algebra; that is to say, dragging makes relationships of logical inclusion between algebraic varieties accessible to pupils at a perceptual level. The role of *lieu muet* in the dynamic of ascending/descending control supports and helps students to produce abductions and provokes the switching between ascending and descending control modalities. A *lieu muet*, as a perceptual counterpart of the above algebraic relationship, *expresses an abduction in a figural and perceptive way*: *C* is indeed the 'rule' which the figures F(P) are the case of, provided the functional dependencies among the constructed objects. In fact, the successive dynamics of pupils' actions have the same structure as those in pencil and paper environments: namely, first the pupils formulate a conjecture in a conditional way (which is a regularity produced by the *lieu muet dragging*), then they make explorations and constructions to validate the hypothesis, as we have seen in the protocols above. The latter are ruled by a descending control; the function of dragging changes: it is now used as a *test* for validating the hypothesis. This dragging dialectic makes accessible a 'jeu de cadre', in the sense of R. Douady between Euclidean geometry and algebraic varieties. The former becomes explicit for pupils through constructions and *dragging test* ruled by descending control; the latter remains implicit, at the perceptual level of *lieu muet dragging*, but the dragging test makes accessible abductions (and possibly conjectures and proofs), which concern more difficult problems than those that they can tackle in pencil and paper environments to pupils.

4. Some open problems.

In our opinion, four main questions seem worthwhile studying: (i) designing didactic situations, where the switching transition by *lieu muet* becomes object of teaching and does not depend only on the ingenuity of some pupils; (ii) exploiting the algebraic aspects incorporated in Cabri, more than the purely Euclidean ones; (iii) applying our model to focus the switching features among the different control modalities within other media used to approach geometry, e. g. 'geometric machines' (Bartolini Bussi, 1993); (iv) using all the discovered variables to define in a suitable way the notion of *didactic space-time of production and communication* for geometry, as we have already done for algebra (Arzarello et al. 1995).

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THE CO-CONSTRUCTION OF MATHEMATICAL KNOWLEDGE: THE EFFECT OF AN INTERVENTION PROGRAMME ON PRIMARY PUPILS' ATTAINMENT[1]

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Abstract

This project explored teaching strategies and learning outcomes with low-attaining 8 year-olds. The research team collaborated with teachers to develop teaching strategies through observing 'real-time' practice using a one-way mirror. The teachers then used these strategies in school. Changes in attainment for the targeted pupils were compared with a matched control group. Targeted pupils substantially out performed control pupils in post intervention assessment in terms of numbers of items correctly answered. Targeted pupils also demonstrated gains over control pupils in terms of developing more effective strategies for answering questions correctly, as demonstrated through item by item analysis of strategies used to answer questions.

1 Theoretical background

Research suggests that two aspects of mental mathematics–known facts and derived facts–are complementary. Studies of arithmetical methods used by 7- to 12- year-olds demonstrate that higher attaining pupils demonstrate the ability to use known number facts to figure out other number facts (Gray, 1991; Steffe, 1983) .

For example, a pupil may 'know by heart' that 5 + 5 = 10 and use this to 'figure out' that 5 + 6 must be eleven, one more than 5 + 5. At a later stage, a pupil may know that 4 x 25 is 100 and use that to figure out that 40 x 24 must be 960.

The evidence suggests that pupils who are able to make these links between recalled and deduced number facts make good progress because each approach supports the other. Eventually, some number facts that pupils previously deduced become known number facts and, in turn, as their range of known number facts expands so too does the range of strategies that they have available for deriving facts.

However, it is also clear that there are many children who, even by the end of primary school, rely more on procedures such as counting to find the answer to calculation and do not make as much progress.

Traditional models of remediation programmes in numeracy in the UK tend to concentrate on the inculcation of arithmetical 'facts' in the expectation that establishing a core of basic knowledge will lay the foundation for later understanding and application. Research findings show this to be inadequate in terms of a view of numeracy as the possession of an integrated network of knowledge, understanding, techniques, strategies and application skills concerned with numbers and number relations and operations.

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Other programmes, for example the Cognitively Guided Instruction work (Carpenter & Lamon, 1988), focus on building up teachers' understanding of pupil networks of knowledge and matching teaching to this.

While the approach we proposed drew to a certain extent on such research findings we were more concerned to work with the teachers on developing a general orientation towards teaching mathematics that acknowledged that learning was a joint responsibility between teacher and pupil (Lave, 1993; Rogoff, 1990; Wertsch, 1985) and which challenged notions such as 'readiness' for particular mathematical ideas .

Questions that the project addressed were focused around examining the effect of the programme on:

- pupils' performance on standard tests;
- pupils' performance on a specially devised diagnostic assessment interview;
- pupils' pre and post intervention strategies as assessed through the interview;
- teachers' attitudes, beliefs and practices in teaching mathematics.

A further focus on inquiry was to explore the nature of training that involved the use of a one-way mirror to observe 'real time' teaching. The New Zealand Reading Recovery Programme (Clay, 1993) had demonstrated this to be a particularly effective method of training the teaching of reading (Clay & Cazden, 1990) and we were interested to examine the extent to which this method might also be powerful in mathematics education.

This paper concentrates on the pupil learning, particularly the quantitative results. Other aspects of the research are discussed in Askew, Bibby & Brown (1997).

2 Methods

Year 3 classes (8 year-olds) in twelve English primary schools constituted the research sample with six of the 12 schools being identified as project schools, the other 6 as control schools. Eight children from each Year 3 class were selected by their teachers. Thus 48 project pupils and 48 control pupils were selected.

These 96 children were identified as being low attainers in mathematics, defining this for practical purposes to be pupils assessed as operating below or just below the expected level of attainment for their age as specified by national tests. The emphasis was on selecting children who were considered to be low attainers in mathematics rather than having special educational needs in mathematics.

The six teachers from the project schools were released for one day per week for twenty weeks over the Autumn and Spring terms 1995/96. In the first term, the teachers started by focusing on the use and interpretation of diagnostic interviews. In the mornings they worked intensively with their group of targeted pupils in their own schools, in two sub-groups of four. In the afternoons, the teachers came together to discuss the teaching strategies being developed and work on identifying effective intervention strategies. Research findings were used to inform the discussion.

In the second term the major element of the afternoon sessions involved the teachers taking it in turns to work with a group of pupils. These sessions took place at an LEA centre where there was the use of a room with a one-way mirror to facilitate observation. The teachers observed each other teach, identified pupil difficulties and developed strategies that they considered to be effective in dealing with these strategies.

Although it was not our main intention to develop models for working with pupils on a one-to-one basis, a pattern of working emerged that appeared to be particularly effective. The twelve to fifteen minutes that the teachers spent working with individual pupils in the mirror room sessions covered four aspects:

- ***Practising counting skills (2-3 minutes)***
  Pupils would work on counting on, in 2s, 5s or 10s forwards and backwards from different starting numbers. They would also work on subitizing-skills (recognising the number of objects in small collections without counting).
- ***Revising individual known facts (2 minutes)***
  The teachers kept an envelope where they and the pupil recorded what an individual knew in number facts and spent some time reinforcing these.
- ***Building on a known fact (8 minutes)***
  The teacher and pupil worked on deriving number facts from one of the pupil's known facts. This provided the main teaching emphasis for the session.
- ***Working with large numbers or problem solving (2 minutes)***
  The final minutes were spent either exploring what could be derived in terms of large numbers (for example working on what double four hundred must be if a pupil knew double four) or putting the number facts being worked on into a problem context.

3 Results: Pupils' responses–quantitative results

Pupils' progress in quantitative terms was monitored using a framework for charting understanding and a related diagnostic interview. Project and control pupils were assessed twice using the diagnostic interview: early in the Autumn term, 1995 and in the summer term 1996. Figure 1 shows the mean test gains for pupils over this period. As it shows, the project pupils made greater gains than the control pupils in terms of the number of items correctly answered in the diagnostic assessment. This gain was statistically significant at the 0.05 level.

However, a focus of the project was whether or not the project pupils also made greater gains in terms of the methods of calculation used in solving the questions in the assessment. The assessment was therefore designed to not only record whether or not the pupil could correctly answer a question but the method of solution used.

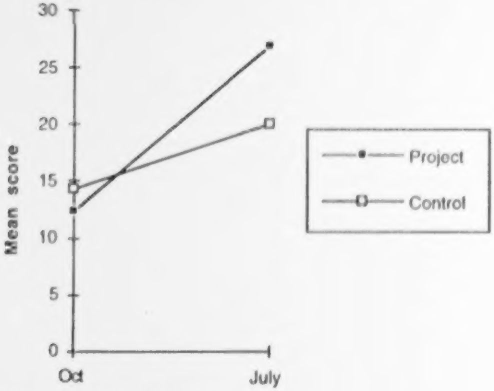

Figure 1: Mean test gains

Pupils' solution methods were coded on the assessment under six different headings, organised in increasing order of sophistication:

- ***Not understood (NU)*** A pupils response was recorded as not understood if she or he could not answer the question through lack of comprehension.
- ***Modelling (M)*** This was used to record if the pupil had used physical objects, including fingers, to answer the question.
- ***Counting (Co)*** If a pupil used a counting on or counting back strategy, without recourse to physical objects, this was recorded as a counting strategy.
- ***Place value (PV)*** Used to code those occasions where pupils used their knowledge of place value and the use of base 10 blocks to answer a question. (This category was not appropriate for all questions.)
- ***Known fact (KF)*** When a pupil answered too rapidly to have used a calculating strategy and indicated that he or she simply knew the answer, this was coded as a known fact.
- ***Derived fact (DF)*** Coding used to indicate that a pupil drew on their bank of known facts to deduce a derived fact.

Every data item on the diagnostic assessment was examined for evidence of changes in strategies. If a pupil made a minor error in finding an answer but the method was correct then this was coded against the method used. However, if a pupil used an inappropriate method or was wildly incorrect, the response was coded as NU.

Figure 2 shows the changes on items which on the first assessment a pupil had not understood. This shows that a proportion of items that were not understood by the pupils on the first assessment were still not understood second time around, but the proportions for project and control pupils are very different. Almost 70 percent of the items that control pupils had not understood in October were still not understood by them in July. In contrast, nearly 70 percent of the items that project pupils had not understood at first were answered using a range of appropriate strategies. These changes are significant (p=0.001).

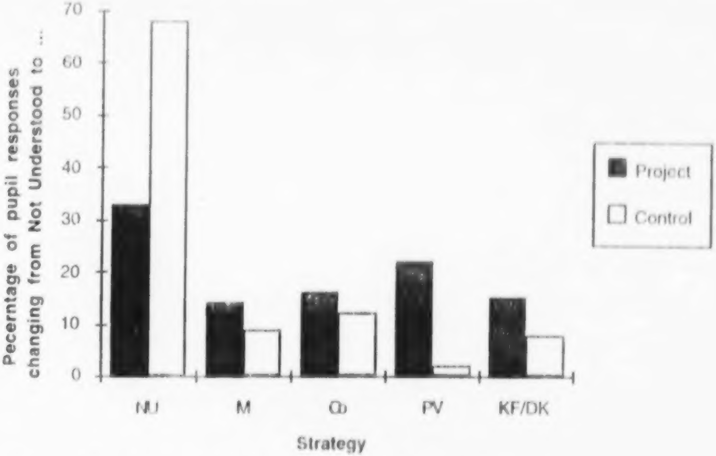

Figure 2: Changes in pupil strategies from Not Understood Oct - July

The range of strategies used by both control and project pupils on items that had previously not been understood spanned modelling through to known and derived facts, but in every category the project pupils out-performed the control pupils.

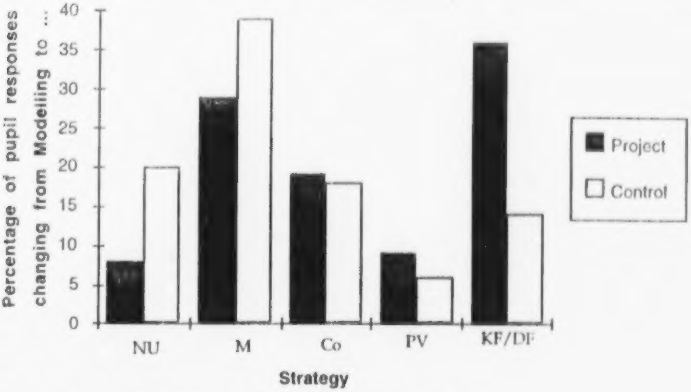

Figure 3: Changes in pupil strategies from Modelling Oct - July

Figure 3 shows the percentage changes away from a modelling strategy. On a number of these items both groups of pupils were still using a modelling strategy at the later date, and, in raw terms, the movement away from this strategy is similar for both groups: around 70 percent of project pupils used a different strategy compared to around 60 percent of control pupils.

However, almost 20 percent of the movement on items for control pupils is accounted for by regression: items which had been answered using a modelling strategy the first time were not understood second time around. The extent of regression on items by project pupils was markedly less, at only around eight percent. Again these changes are statistically significant (p=0.001).

Particularly striking is the changes from a modelling strategy to using known or derived facts. Thirty-six percent of the items that project pupils had originally answered using a modelling strategy were subsequently answered using a known or derived fact. The corresponding figure for control pupils was 16 percent.

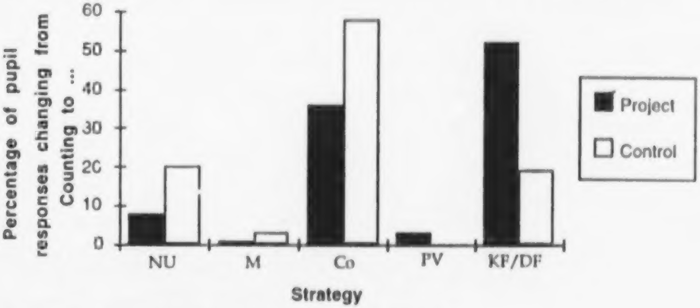

Figure 4: Changes in pupil strategies from Counting Oct - July

Project pupils substantially out-performed control pupils on movement from counting strategies to the use of known and derived facts: 51 percent as compared to 19 percent, as figure 4 shows.

Figure 4 again also shows that in both control and project groups, on a number of items answered using a counting strategy, at the second assessment point there was either no change in strategy or some regression. The figures for the two groups are again markedly different: on 81 percent of the items control pupils had not made any progress in terms of strategies used, compared to 45 percent for project pupils. All these changes are significant (p=0.001).

The data indicates that on both accounts, number of items correctly answered and range of strategies used, project pupils significantly outperformed control pupils.

It had been our intention to also monitor performance at class level. Number items on end national test that the pupils had taken at the end of the previous year were to be analysed item by item and classes reassessed on these same items one year later.

However, in gathering this data it became clear that the range of conditions under which the national tests had been administered had been so varied as to make this data too unreliable. For example, some classes had had every item read out to them regardless of linguistic ability whereas other teachers had only read items to particular groups of pupils. Again, some pupils had been allowed to take as much time over the test as they liked, whereas others had been given a strict time limit.

However, the observation data of classroom practice clearly indicated that the practices the project teachers used with the targeted pupils were adapted to being used with whole classes. It would therefore be reasonable to assume that if the changes in performance for the targeted project pupils were attributable to the teaching practices then there would have been some impact at the class level.

Further to this observation, other similar research has indicated that teaching directly at developing pupils conceptual understanding does not lead to a drop in attainment on more procedural standardised tests (Carpenter, Fennema, Peterson & Carey, 1988; Cobb et al., 1991; Simon & Schifter, 1993)

4 Discussion

The quantitative data analysis gives us confidence that the nature of the intervention was effective in raising pupil attainment. Qualitative data analysis is beginning to provide insight into possible reasons for this effectiveness.

Pupils' progress was also monitored through observation. This was done in normal classroom conditions and also from the data gathered for the small group of pupils used in the mirror room sessions. Field notes and recordings of teacher and pupil discussion continue to be analysed, but some insights that are developing.

In particular the analysis of the qualitative data continues to raise questions about the extent to which low attainment is actually the result of some 'deficit' in the pupil or co-constructed between the teacher and pupil through each not being totally clear about the expectations of the other.

For example, many pupils seemed to be doing what they thought was expected of them, rather than relying on their mathematical understanding. For example, the teachers would often ask pupils to count out, say, 10 cubes. Moments later when asked how many cubes were there, the pupils would re-count them. In discussion, it became clear that the teachers did not discourage this re-counting as they interpreted as either demonstrating that the pupils lacked confidence or that they need to re-inforce their counting skills. However, once the teachers started asking the pupils if they could remember how many there were without counting, the pupils could answer easily. The counting, it seemed, was a response to what the pupils thought the teachers expected them to do, rather than what they needed to do.

The mirror room work was a powerful means of eliciting what the teacher believed were the salient aspects of the teaching and learning. Having to comment as things happened meant that it was harder for the teachers to present observations that they might think were most acceptable. For example, while the issue of pupil 'readiness' is no longer fashionable as a theory in mathematics education, this came through as a strong construct that many of the teachers were still using to frame their teaching. If pupils did not spontaneously demonstrate strategies this was 'read' by the teachers as indicating that the pupils were not yet ready to be taught such strategies. The use of the mirror room not only enabled the uncovering of such expectations but for these to be challenged and the teachers to try out alternative approaches in a supportive environment.

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Dialectical proof: should we teach it to physics students?

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ABSTRACT

This paper attempts to introduce a dialectical conception of proof into the discussion of proof teaching. Such a conception is used to design and interpret results of a teaching experience with physics students where an explaining proof of L'Hôpital's rule that unifies both cases, 0/0 and ∞/∞, *was tested.*

1. For a dialectical conception of proof: circularity and essence

It is generally accepted that proof was born in the fifth century B. C. in Greece. Was it accidental that it was born at the same place and time as democracy and banking credit? Perhaps not. A century earlier, tradition held that, in spite of soil impoverishment, the eldest son should stay at home, taking care of the rituals around the sacred fire and guarding his ancestors' tomb. It was the time of reforms of Solon and Dracon. The younger brothers were sent abroad to commerce; they came back rich and increased the power of the *polis* as opposed to that of landowners. The conflict between tradition and money could not be solved by the sword since it stemmed from inside the families. Democracy became necessary. The *agora*, philosophy and, following the same line, mathematical proof emerged. Three centuries later, when it became necessary to diffuse this genial solution to the whole world, the Greeks built Alexandria's light-house, more a symbol than an useful device. Proofs were arranged in logical packets. It was the time of Euclid.

The dialectical circularity

The year 1976 saw two landmarks for Mathematics Education: the announcement of the computer proof of the four color theorem [Apple & Haken, 1976] and the publication of *Proofs and Refutations* [Lakatos, 1976]. Philosophers of mathematics were thrown into turmoil. Is mathematics fallible? Are computers reliable? Will the computer culture introduce new paradigms of proof into mathematics? At the apex of the stir, Horgan [1993] blatantly announced *The death of proof*. While philosophers of mathematics struggled to reestablish peace, mathematics educators discussed the (new) role of proof in their classroom. This story is nicely told in Hanna [1995, 1996].

From Euclid to Hilbert, proof underwent a long development. Instead of departing from objects and common sense truths about them, objects became "symbolic entities which owe their existence only to the fact that they satisfy the rules by which they are axiomatically linked" [Hanna & Jahnke, 1993, p. 425]. With computers, axiomatical linkage further escaped control, as in zero-knowledge and holographic proofs, [see Hanna, 1996, p. 23].

In the course of the discussion stirred up by the computer issue, every item that had traditionally been evoked to present proof as a guarantee of truth was challenged. Hanna [1983, quoted in Neubrand 1989] makes an effort to characterize conditions by which mathematicians accept a new theorem. The new result should *be understandable, significant*, and *consistent*, the author should have an *unimpeachable reputation*, and there should exist a *convincing argument*. Of these five factors, Neubrand stresses the last one: "It is somewhat like a *sine qua non* condition and should therefore head all the other social factors" [Neubrand, 1989, p. 6]. David Hersh reinforces the "convincing" factor and introduces the community of "judges". "In mathematical practice, in the real life of living mathematicians, proof is *convincing argument, as judged by qualified judges*" [Hersh, 1993, p. 389]. "At the stage of creation, proofs are often presented in front of a blackboard, hopefully and tentatively" [ibid. p. 390].

These arguments stress the conception of mathematics as a *discursive practice* [McBride, 1989]. "With few exceptions, mathematicians have only one way to test or "prove" their work – invite everybody who is interested to have a shot at it. So the day-to-day mathematical meaning of "proof " agrees with the colloquial meaning" [Hersh, 1993, p. 392]. One could be tempted to say that proof is just a way of speaking, a form of speech of a certain community: "What mathematicians at large sanction and accept is correct" [Hersh, 1993, p. 392]. Pushing such a shift towards a pragmatic view a little further, we would infer that mathematics is an office room conspiracy of scientists. Scared by this conclusion, we would go back in search of a new and stronger normative attitude. In order to stop swinging back and forth, we only have to assume the dialectical circularity in its sharpest form: *a theorem is true because mathematicians say it is; but they would not say it is true if it were not.* Now we can move on.

The movement of the concept: essence and *dasein*

"The movement is the double process and unfolding of the whole; thus each moment places the other at the same time and each one has both moments in itself as two aspects; taken together, these aspects constitute the whole, insofar as they dissolve themselves and make themselves moments of such a whole" [Hegel, 1941, p. 36, my translation].

Throughout the literature about proofs, there are at least three consensual points: 1) proofs have to do with the general idea of *truth* (convincing, explaining, justifying, demonstrating, deducing, etc.); 2) we should certainly teach proofs; 3) formalization is not the best way to teach proofs.

From the perspective of the person who is reading them, formal proofs have long been sharply criticized: "What one generally gets in print is a daunting cliff that only an experienced mountaineer might attempt to scale and even then only with special equipment" [Epstein & Levy, 1995, p. 670].

"The proof follows a course that starts at an arbitrary point, so that one cannot know the relation between this initial point and the result that must come from it. The proof's bearing requires such determinations and such relations and discards others, so that one cannot immediately realize under which necessity this happens; an exterior finality commands such a movement" [Hegel, 1941, p. 37, underlining added].

As a consequence of such criticism, attempts have been made to distinguish aspects of formal proofs capable of providing alternative approaches to be used in classrooms. Many categories have been proposed:

Explanation, proof and *demonstration* [Balachef 1987], *preformal* versus formal proofs [Blum & Kirsch, 1991], *proofs that prove* versus *proofs that explain* [Hanna, 1995], *formal* versus *intuitive* proofs [Fishbein, 1982], *proofs to try and test* versus *proofs to establish beyond doubt* [Epstein & Levy, 1995], *analytical* versus *substantial arguments* [Godino & Recio, 1997], *structural* versus *linear-styled* proofs [Alibert & Thomas, 1994], *analytical*, *empirical* and *external* proof schemes [Harel & Sowder, 1996], *humanist* versus *absolutist* mathematics teacher [Hersh, 1993], *technical* versus *critical* perspectives [Garnica, 1995].

All these attempts point to a subjacent aspect of proofs that lies beside or underneath the pure statement of a theorem and its final written form. This aspect hinges on the above-mentioned *discursive practice* that we can now identify with what Hegel calls "exterior finality". In the following paragraphs, I shall underline the specific references to the exterior finality that rules the development of proof:

"Should we give the impression that the best mathematician is some sort of magic conjured out of thin air by extraordinary people when it is actually the result of hard work and of intuition built on the study of many special cases?" [Epstein & Levy, 1995, p. 670].

"(...) a 'convincing argument' is not simply a sequence of correct answers. One always expects some 'qualitative' reason or an intuitive capable basic idea behind the – nevertheless necessary – single steps of the proof" [Neubrand, 1989, p. 4].

"The feeling of the universal necessity of a certain property is not reducible to a pure conceptual format. It is a feeling of agreement, a basis of belief, an intuition – but which is congruent with the corresponding formal acceptance" [Fishbein, 1982, p. 17].

"The best proof is one which also helps mathematicians to understand the meaning of the theorem being proved: to see not only *that* it is true but also *why it is true*" [Hanna, 1995, p. 47].

"The concept of formal (...) proof can become an effective instrument for reasoning process if, and only if, it gets the qualities required by adaptive empirical behavior" [Fishbein, 1982, p. 17].

"(...) the general plan is never revealed (...) and the student may be reduced to merely checking the validity of the deduction at each step" [Alibert and Thomas, 1994, p. 222].

What do these authors mean by "general plan", "adaptive behavior", "feeling of agreement", etc.? What are they pointing at? What are they looking for? Hegel would bluntly call it *the essence*. The "formal proof" from which they are trying to distance themselves, Hegel would call *dasein* ("l'être-là").

"Also in philosophical knowledge, the development of *dasein* is different from the development of the essence or of the inner nature of the thing" [Hegel, 1941, p. 37] .

What Lakatos [1976] describes is the development of essence. In brief, if we assume the dialectical principle that all determinations are relative to each other, we can consider the conceptual movement started in 1976 as the development of a single whole, along which distinct aspects of proof are separated from each other. While we think of proof as a fixed pivot around which we have been turning, proof is actually constantly becoming everything that we have been saying about it.

2. The study

The study was carried out in a one-year freshmen calculus course for physics students at UNESP, Rio Claro, SP, Brazil during 1995-97. Approximate numbers for each year have been: 60 students enroll (40 freshmen plus 20 repeaters), 40 attend classes, 20 pass. Half of the students have part-time jobs and eighty per cent live in nearby cities.

The Campus remains empty during weekends. The syllabus covers the first volume of the textbook: Swokowski [1983].

Early in 1995 the Physics Department made the following request of teachers in charge of mathematics courses for the physics students: "We need mathematics as instrumentation for physics. We would like the students to gain familiarity with the textbook. We would like you to teach *less theorems and proofs* and more exercises and applications to physics". A negative reference was made to the linear algebra course where questions like "show that $x.0 = 0$" used to appear in the exams.

What should I have done ? Should I have ignored the request and assumed that my mission would be to open a window through which students would have the opportunity to contemplate the mathematical world during some time, before they proceed in their curriculum? Should I teach proofs in such a context? As far as I know, the question of teaching proofs in mathematics courses for service departments has not been addressed in the literature. Specifically, *should we teach mathematical proofs to physics students?* If so, why and how? This question is not trivial since, as we have seen, mathematical proofs have to do with mathematical truth while physics students are being trained to abide by criteria of truth specific to their science. "If you believe, as many do, that proof is math and math is proof, then, in a math course, you're duty bound to prove something" [Hersh, 1993, p. 396]. Well, I do not believe so. As the new calculus teacher for physics freshmen in 1995, I took the physics department's request as a challenge, not as an interference.

Here is what we [1] did . During 1995, we followed suggestions of Alibert & Thomas 1994, Arsac et al [1992, and Legrand, 1990]. We tried to introduce scientific debates into the classroom. We started with graphical problems about kinematics and tried to gradually introduce mathematical instrumentation as problems became more algebraic. Students were exhorted not to use formulas or results unless they could justify them. Exercises were taken from the textbook and proofs were introduced through worksheets according to the belief that "the main function of proof in mathematics education is surely that of explanation" [Hanna, 1995, p. 47] and that the mathematics teacher should "use the most enlightening proof, not necessarily the most general or the shortest" [Hersh, 1993, p. 397]. The approximately forty students were generally organized in groups of four. Slow learners were invited to extra tutorial sessions once a week. [See Baldino, 1997].

We never succeeded in keeping more than one fourth of the class interested in the debate. In the beginning of next year, the Physics Department complained that the students that we had passed on to them were poor calculators of integrals. Therefore, during 1996 and 1997, the course was split: four hours a week were dedicated to concepts and applications and two hours a week were dedicated to straightforward calculations of limits, derivatives and primitives. Scientific debate was restricted to tutorial sessions. Now the students were told that they would eventually receive worksheets with justifications of the results that they were already using, such as the chain rule and the fundamental theorem of calculus.

L'Hôpital's rule emerged spontaneously from the classroom culture, introduced by those repeating the course. Proofs that we found in the literature could not be classified as explaining proofs. Besides, the case ∞/∞ cannot be immediately reduced to the case 0/0, unless the existence of the quotient limit of the functions can be granted beforehand.

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[1] Myself and two graduate students: Tania Cabral and Ronaldo Melo.

Having to show the existence of this limit makes the proof considerably harder. So, we decided to take up Hanna's challenge: "Unfortunately there is no guarantee that every theorem we might like to use will have a proof that explains" [Hanna, 1995, p. 48

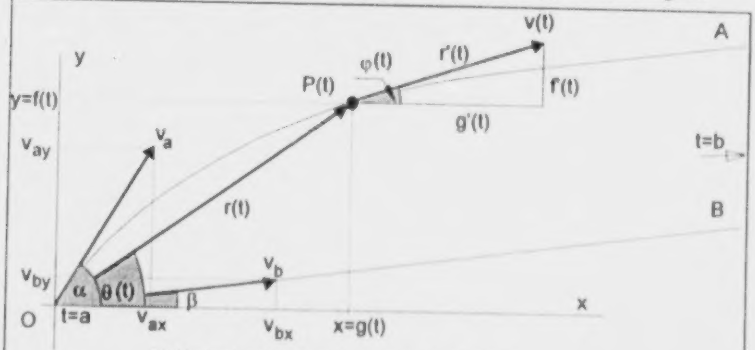
Textbooks suggested in the course's syllabus adopt different strategies in order to circumvent the difficulty with the ∞/∞ case. Hughes-Hallet et al. [1994]] do not even mention L'Hôpital's rule. The authors evoke graphical calculators to solve classical limits and prove the error expression in Taylor's formula by successive integration. Other books do not mention that this case is more difficult: (Carvalho e Silva [1994, p. 279], Ayres Jr. [1981]). Most authors mention the case ∞/∞ but omit the proof and send the reader to "more advanced texts": Linch et al. [1973, p. 514], Apostol [1976, p. 300], Leithold [1976, p. 510], Keisler [1986, p. 246], Swokowski [1983, p. 622]. Shenk [1979, p. 304] leaves the proof to an honors exercise, Strang [1991, p. 153] gives a proof assuming the existence of the limit, and Simmons [1985, p. 569] gives a hint for the proof. Spivak [1967, p. 186], Piskunov [1977, p. 149] and Seeley [1968, p. 643] are among the few that present the proof in detail; unfortunately these are epsilontic proofs.

Seeley also offers one figure. We evaluated that this figure contained the essence of the argument and took it as a starting point to design a worksheet appropriate for physics students. In so doing, we were guided by directives summarized in the following table of oppositions.

Deductive proof	*Dialectical proof*
Development of *dasein*	Development of essence
Hypothesis - thesis – demonstration	Thesis – demonstration - hypothesis
Linearity of the significant chain	Network of models
General case first	Particular case first
Primacy of concept definition	Primacy of concept image

3. Worksheet: Why does L'Hôpital's rule work?

Consider a moving particle along the trajectory AO in the xy-plane. Suppose that at instant t, the particle is at P(t) = (x(t), y(t)) with position vector r(t), as in the figure.



L'Hôpital's rule: development of essence

Suppose that at instant t = a the particle is at the origin and as t tends to b the particle gets away from the origin so that both its coordinates tend to infinity. Mathematically we are saying that: $\lim\limits_{t \to a^+} \frac{f(t)}{g(t)} = \frac{0}{0}$ and $\lim\limits_{t \to b^-} \frac{f(t)}{g(t)} = \frac{+\infty}{+\infty}$. Let the particle's velocity at instant t be $v(t) = r'(t) = (g'(t), f'(t))$. Suppose that at t = a the velocity has an initial value v_a and that, as t tends to b, the velocity tends to a final value v_b parallel to the straight line OB. Let $v_a = (v_{ax}, v_{ay})$ and $v_b = (v_{bx}, v_{by})$ be the components of the initial and final velocities. Let the angles $\varphi(t)$, $\theta(t)$, α and β be as in the figure.

1. Describe the trajectory as t tends to b.
2. Fill in the blanks:

$$\lim_{t \to a^+} \varphi(t) = \ldots \quad \lim_{t \to a^+} \theta(t) = \ldots \quad \lim_{t \to b^-} \varphi(t) = \ldots \quad \lim_{t \to b^-} \theta(t) = \ldots$$

3. Consider the slopes of the straight lines determined by the vectors r'(t), r(t), v_a e v_b and fill in the blanks:

$$\lim_{t \to a^+} \frac{f'(t)}{g'(t)} = \ldots \quad \lim_{t \to a^+} \frac{f(t)}{g(t)} = \ldots \quad \lim_{t \to b^-} \frac{f'(t)}{g'(t)} = \ldots \quad \lim_{t \to b^-} \frac{f(t)}{g(t)} = \ldots$$

4. Conclude: why does L'Hôpital's rule work?

Two other sheets with different and higher degrees of formalization were distributed to the students together with this one.

4. Outcomes and discussion

The worksheet was introduced to the students early in June and repeated in early November. In each case we asked for a report: "Explain why L'Hôpital's rule works". Of course, I expected the students to say *Ah ha! Now I know why I am calculating limits in this way*. Interestingly enough, the hard point in formal proofs did not seem to hinder them: all groups could describe reasonably well that, as t tends to b, the trajectory tends asymptotically to a straight line parallel to OB. The existence of the quotient limit of the functions was proved *in action*. Only one group needed help, and that was supplied by hand-waving and dragging an eraser on the table.

I made an attempt to analyze the students' protocols in terms of *proof schemes* proposed by Harel & Sowder [1996]. It seems clear that the proposed explaining proof may be classified as a *transformational proof scheme*: "justifications attend to the generality aspects of a conjecture and involve mental operations that are goal oriented and attended-anticipatory" [ibid. p. 62]. Many protocols clearly indicate an *authoritarian proof scheme*. For these students, L'Hôpital's rule authorizes them to use a procedure either to get rid of the indetermination or to proceed when one gets stuck. "It holds because calculating the quotient of the functions is the same as calculating the quotient of their derivatives". Other protocols indicate a *symbolic proof scheme*: indeterminations are puzzling objects that possess a life of their own, and L'Hôpital's rule explains exactly who such entities are: "When we get $\frac{0}{0}$ or $\frac{\infty}{\infty}$, this means that the limit of $\frac{f}{g}$ is equal to the limit of $\frac{f'}{g'}$". Other protocols evoke examples and may be included as *empirical proof schemes*: "We apply L'Hôpital's rule to impose continuity on the indetermination, as in $\frac{\sin x}{x}$".

However, not a single group was able to clearly reproduce the unifying argument: slopes of secant and tangent lines tend to the same value. So, how should we rate our didactical effort? Flat failure? Can we say that we wasted four hours of class-time? Perhaps not, if we look more closely at some protocols such as this one:

"L'Hôpital's rule holds because, when we apply the quotient rule and we find an indeterminate form, we apply L'Hôpital's rule that proves that, given a point P, when such a point tends to zero, $f'(t)$ and $g'(t)$ tend to the initial values v_{ox} and v_{oy}, where the angle φ tends to the value of α."

That is: "L'Hôpital's rule holds because it proves a property about trajectories". This seems to nicely reproduce, at the level of pedagogy, what Hanna & Jahnke call "appeal to the future" at the level of history. Like in the case of Newton's derivation of Kepler's laws from the Law of gravitation, "that which is proved serves to legitimize the assumptions from which it is derived" [Hanna & Jahnke, 1993, p. 428].

Other protocols are quite difficult to interpret from the point of view of deductive proof. "L'Hôpital's rule holds because, when we apply the limit in indeterminate forms, the function tends to different "angles". However, the slope, when it tends to zero, is the same. Deriving the function, we raise the indetermination and we can find the limits if they exist." A perfect salad! However, if we accept dialectical circularities and obscurities, we may look at this student's development as if we were looking to a developing photograph in a dark room: the picture appears evenly all over the cardboard, not from top to bottom or from left to right. Good mathematicians also "develop" themselves in this way. If a calculus student talks about "limits of infinitesimals" we would take it as a symptom of confused ideas. Nevertheless: "The determination of the tangent to the curve is reduced to the determination of the limit of the ratio of two infinitely small quantities." [Duhamell, 1874, p. 91].

As a final word, we would say that reduction of proof to *deductive proof* is a one-sided view. It may be a necessary ideology for mathematicians' daily scientific practice of theorem-proving, but it does not suffice for mathematics education. "Dialectical view of proof" is an expression borrowed from Hanna & Jahnke, [1993, p. 422]. However, the concept can be traced back to Hegel. *Dialectical proof* is a concept intended to apprehend the development of History and of human subjects in a single unity: the *movement of concept*.

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Lacan and the school's credit system

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Abstract

In his plenary address at PME-21, Shlomo Vinner referred to the school system as being above all, a *credit system*. He introduced the notions of *pseudo* and *true knowledge* and hinted at the dimension of faking and deceiving inherent to educational actions. From this perspective, we characterize a *mismatch* between students declared intentions and their actions. We briefly discuss the difficulties of considering this mismatch phenomenon either from the strict cognitive point of view or from the perspective of Peirce's semiotics. We offer an alternative approach based on Lacan's psychoanalytic theory. We introduce the four types of discourse conceptualized by Lacan. We invest *the master's discourse* to analyze a hypothetical *traditional* classroom. The other three forms of discourse, the *university's* the *object's* and the *hysteric's discourses* are presented schematically.

Introduction

The plenary conference of Shlomo Vinner in PME-21 [Vinner, 1997] pointed out a phenomenon that has not yet been adequately considered, if it has been considered at all, in the literature about mathematics education. Issues concerning the fact that *"we cannot avoid dealing with values when we teach"* [p. 69] have long been addressed by authors, specially of the Ethnomathematics group [see Powell and Frankenstein (Eds.), 1997]. However, issues stemming from the recognition *that "the educational system is, above all, a credit system"* [p. 68] have not been dealt with. In the available extensive literature about assessment, there is no reference to the credit system [see Gomes da Silva's 1997 extensive bibliographic review].

Recognition of the existence of the credit system in school leads Vinner [1997] to introduce the dimension of faking and deceiving, both on the part of the student as well as on the part of the teacher. He admits that the student may develop a certain knowledge about *"how to get credit from the educational system"* and he calls that *"pseudo knowledge"*, as opposed to *"true mathematical knowledge"* which is *"knowledge desirable by the educational system"* [p. 68]. Vinner warns that the teacher can be easily deceived by this pseudo knowledge. Then he adds: *"But don't we want to be deceived, especially when it comes to our student's achievements?"* [p. 73]. In one word, Vinner reveals that the widespread mismatch between discourse and action is also present in the mathematics classroom: the student declares that s/he wants to learn, but all her/his actions demonstrate that s/he wants to pass. We shall call this the *mismatch phenomenon*. This phenomenon has been considered by Cabral [1992] during a one-year observation of a calculus classroom. See also Cabral [1998].

Since pseudo-knowledge is involved, the mismatch phenomenon cannot be approached from a strict cognitivist and formalist framework. *"From this perspective, cognitive processes are viewed as pure forms while the environment is factored out as variables only tangentially related to cognitive events"* [Meira, 1997, p. 232). This author points to the necessity of considering issues of *negotiation of meaning* within the classroom culture. In fact, meaning production and semiotics are becoming an increasingly important issue in PME. After Walkerdine's [1988] pioneering work, the subject has been taken up, for instance by Radford & Grenier [1996] and Vile & Lerman [1996]. In PME-21, questions of meaning production were specifically addressed by Godino and Recio [1997], James, Kent & Noss [1997], and Meira [1997]. A discussion group was formed [Radford & Vile, 1997] and probably a working group will come into existence very soon.

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[1] Partial support from CAPES.

In summary, according to Vile & Lerman [1996], semiotics has two "roots", one in the work of C.S. Peirce and the other in the structuralism through the work of Saussure and Barthes. The vast majority of research in mathematics education adopt the view that *language is created by human beings* with the purpose of communication. In this line we should include researches that seek support on Piaget and Vygostky, such as the ones developed by the Advanced Mathematical Thinking group of PME and, in particular, former works of S. Vinner. Development of this line assigns a descriptive role to language and leads to Peirce. The other root of semiotics assigns a constitutive role to language: *reality and human beings are creations of language*. Here we should include Vinner [1997] insofar as it assigns a determinant role to social interaction (credit system). Instead of

human beings→social interaction→communication needs→language

we have:

social interaction→language→human beings→communication

The aim of this paper is to briefly discuss the mismatch phenomenon in the framework of Peirce's semiotics and to suggest an alternative approach from the Freud-Lacan perspective. One root of this perspective certainly lies in Saussure's semiotics; we contend that another one is to be found in Hegel and Marx.

From Peirce to Lacan

When a student declares that s/he wants to learn but all her/his actions demonstrate that s/he mostly wants to get credit, s/he is contributing to the classroom culture in a way that peers may imitate. This is an "act of communication". The *mismatch* may be considered as a semiotic action that fits Peirce's definition of sign: *"something which stands to somebody for something in some respect or capacity"* (Peirce quoted by Vile & Lerman, 1996, p. 396). Difficulties start, however, when we note that *"according to Peirce, a sign captures only an aspect of its* object, *this aspect is the* ground *of the sign, that is, a component of the signified* (signifié) *associated to the object* (Radford & Grenier, 1996, p. 179). What would the *object* and the *signified* be in this case? What is the communication act? "When one person catches another's *idea* the two are obviously engaged in communication, furthermore they will "catch each other's ideas" because of a belief in a shared understanding" [Vile & Lerman, 1997, p. 398]. We shall return to this quotation below.

Rather than force Peirce's semiotics into a domain that it was not primarily intended to account for, we shift to the perspective of Freud-Lacan. For Lacan, both, declaration and action are signifiers, and it is precisely in the mismatch between signifiers that he places *desire*, the pivot around which the psychoanalytical movement turns. In this theory, the mismatch phenomenon is a special case of a general and unavoidable language mismatch, a vicissitude of any talking being. The joint occurrence of the signifiers "mathematics" and "psychoanalysis" is rare and recent, both in mathematics education and in psychoanalysis. There have been numerous attempts to look for support in psychoanalysis to solve problems of anxiety or motivation in mathematics teaching and learning. Such attempts invariably seek some kind of *improvement* and place themselves under the shield of a certain *hope*. [See the specific edition of For the Learning of

Mathematics, Blanchard-Laville, 1992.] However hope has to be put momentarily aside if we hope to say anything structurally rigorous.

For the sake of the argument we discard the academic conception that it is necessary to first acquire an overall view of Lacan's work in order to be able to speak about it. Indeed, what stand could support such a view? At the moment that we uttered the first statement, we would already be in the plane of *enunciation*; that is, we would be inside the domain of language, hence inside the reality that psychoanalysis takes as object. Therefore *we would be obliged to listen to what Lacan's theory has to say about what we would be trying to say about it*. We are inside the reality that we would be trying to contemplate, or to apply from the "exterior". In order to circumvent this difficulty, we prefer to take a fragment of Lacan's work, *Le Seminaire 17*, and use it to analyze a hypothetical classroom situation.

According to Lacan, discourse should not be understood as a word-flow emitted by one subject (the teacher) and listened to by another (the student). For Lacan, the concept of discourse is precisely this: a *statute of statements*. In order to consider what happens when someone talks, Lacan discards all notions from communication theory (transmitter, receiver, code, noise, etc.). The phenomenon of speech is much more complex, because when the subject talks, s/he becomes subjected to a complex process involving a *double structural mismatch*. On the one hand, the intended meaning of what s/he says escapes the control of the subject. S/he has to wait for a response through which the interlocutor will inform her/him what has in fact been understood by what s/he said. Perhaps more than anyone else, teachers know this vicissitude. On the other hand, even if the interlocutor tries to repeat word for word what the subject has said, some difference is introduced simply because the phrase is uttered a little later, by somebody else, with a different accent, etc. The meaning intended by the subject is not received back exactly as s/he had hoped. A metaphor is spoken and a metonym is heard back.

The discourse is the norm of what fits and what does not fit into the Other's ears, and consequently what can and what cannot be said by the speaker. Lacan denotes the "Other", written with a capital letter, "A", from the Latin "alter", and calls it the big-Other. It is the frame that circumscribes the speaker. The big-Other is determined by the language, by the historical moment, by the culture of the social formation where conversation takes place. It includes all possible signifiers available to the speaker, as well as the rules to use them. It contains the dictionary, the grammar and the laws

Due to the vicissitude of the speaker having to wait for the Other in order to learn the meaning of what s/he says, Lacan denotes the subject by a barred S ($\not{S}$). Due to the impossibility of the Other fulfilling the subject's hopes for understanding, Lacan denotes the Other by a barred A ($\not{A}$). The mismatch is denoted $\lozenge$. Hence, the whole of Lacan's work can be condensed in a single formula, $\not{S} \lozenge \not{A}$, the dialectics of the Subject and the Other.

Following Lacan, we would say that the discourse in a classroom should be understood as a *joint effort* of students and teacher in order to *sustain* a certain relation (the statute) of actions and utterances; it is an effort to cover up for the necessary language mismatch. When such a cover-up succeeds, people used to say that they have "communicated". We now go back to the above quotation: "When one person catches another's *idea* the two are obviously engaged in communication, furthermore they will "catch each other's ideas" because of a belief in a shared understanding" [Vile & Lerman 1997, p. 398]. We should say that it is not because of the "belief in a shared understanding" that people will "catch each other's ideas". It is because they *make an effort to share each other's ideas* that they can *catch each other's beliefs*. Through the communication *efforts* (not acts !) people end up believing that a pre-existing "shared belief" was responsible for their communication. In one word, for Peirce, no shared beliefs implies no communication, whereas for Lacan, the effort of communication implies the development of shared beliefs.

The demand: every discourse is an answer

Discourse does not start only when the invited speaker says "Ladies and gentlemen". Well before this inaugural moment, people got dressed, left their homes and drove to the conference site attracted by its title, by the speaker's name, by an invitation. They sat in the room looking at the lighted pulpit supporting the microphone. What can be said there is already determined by the expectation imposed by the situation. Hence, *the discourse is always the answer to a demand*. In such a situation Michel Foucault said he would have liked to have had a nameless voice precede him in such a way that he could just intercalate his words in its moments of silence. We conclude that it is not only the question that determines the answer, but that *the question itself already is an answer*.

So, the discourse starts before the speech; it starts with the demand. However, paradoxically, the demand is only complete when it obtains an answer. When, to the astonishment of all, Cicero started his discourse shouting about Catilina, instead of the normal form of the Roman Senate, he defined the seriousness of the political tension of the moment, and in this way, he determined the demand to which he was answering. Hence, speech and demand determine each other. Rather than being a "communication" or a "message", the speech is what decides which expectation was set up to listen to it. It is the speech that decides what the demand was. It is the answer that determines the question. The discourse depends on the spoken word but is not reduced to it.

Speaking is a complex process involving simultaneously three registers: *imaginary* (pre-suppositions of action), *symbolic* (language), and *real* (*jouissance*[2]). Discourse requires a choice among possible utterances according to the directive lines that interlocutors engaged in the discursive situation struggle to maintain. Systematic errors are difficult to eradicate because the choice of response that leads to them is of the order of the real; it involves the subject's *jouissance* organization. We contend that we need to introduce into mathematics education a theory that takes into consideration that the answers that our students give us cannot be listened to only from the cognitive point of view, because students are much more than knowing subjects; they are *desiring subjects*.

Lacan's four discourses: the master's discourse

For the sake of the argument we have assembled all "negative" traces of classroom culture in a single exaggerated cartoon, labeled *traditional teaching*. Probably such a classroom cannot be found in its entirety anywhere. Each real classroom has some traces of it. We shall consider such a classroom under Lacan's concept of discourse. Traditional teaching consists in a sequence of four moments: 1. An inaugural moment, based on plain authority, the course's introduction. This is the *master's discourse*. 2. A second moment dominated by a verbal flux emanating from the teacher, the *scéance magistrale*. This is the *university's discourse*. 3. A third moment centered on the credit system when the student has to choose between two strategies or two objects, *pseudo* or *true* knowledge. This is the *object's discourse*. 4. A final moment where the student only hopes for luck in the exam. This is the *hysterics' discourse*.

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[2] We keep the French word in italics since the English correspondent term "enjoyance" is not in the dictionary

In fact, most of the literature on Mathematics education, including Baldino [1997], suggest that these discourses should occur in the reverse order in the classroom. From the inaugural master's discourse, we should move to a form of hysterics' discourse when the student has to face some sort of unbalance. Next, we should provide the opportunity for an object's discourse centered on true knowledge, not on the credit system. Finally, the university's discourse should be promoted, this time with the student occupying the place of the speaker. At the beginning of the school year, the stirring students are waiting for the teacher. Somebody enters the room and announces: - *I am the mathematics teacher*. The introduction is gratuitous, since it is time for the mathematics class, he is much older than the students, and he places himself behind the desk on which he lays his briefcase. No need to say anything... After a certain time, the students get up and go out. The class is finished. The initial and final moments are the marks of a certain duration. They appear as void and irrelevant. However, the teacher's leading presence in itself announces the speech that is going to take place: *the mathematics class*.

Before the redundant introduction mentioned above, the teacher's figure is a pure signifier, without a meaning, or to say it better, is a signifier whose meaning falls on itself. The presence of somebody whom the students can consider "the teacher" is necessary in order to start the game. At this moment, the teacher's figure is no more than this necessary mark, supported by the institution's insignias: the desk, the stage, the blackboard... The teacher is not yet a subject of the process that is going to take place. The king's signet ring is the clearest possible example of such a primary signifier. Its mark means nothing, but without it on the sealing-wax, no document is legitimate, no order is obeyed. The master's signifier is $\mathbf{S_1}$.

When the teacher enters the room, some students sit down. Others go on playing. From this moment on, their playing has the connotation of a challenge, a test of the teacher's patience and resistance. The exercise of certain powers is the way to keep them active. Michel Foucault teaches us that it is via its exercise that power constitutes itself. Bourdieu and Passeron teach us that the more power disguises itself, the stronger it becomes. Such considerations lead us to understand that the role of the nasty student is not so undesirable as it is generally thought, but it is necessary to invigorate obedience. Therefore, it is by the investment of a certain knowledge, precisely of a know-how, that at this inaugural moment the students make a meaningless signifier out of the teacher, in the name of which they demand the strengthening of a certain relation (statute) among actions and utterances between them and the teacher. The teacher is supposed to stop their playing in the name of the official knowledge cast in the syllabus and "get the class started". The signifier of knowledge is $\mathbf{S_2}$.

From the moment of his introduction, it is well known that the teacher will have obtained authorization to exercise his functions. It is also known that such an authorization is based on a certificate or degree that testifies his success in the school credit system. Now, the students know the pathway of promotion in this system better than anybody else. They know that in the school system, most of the time one passes without knowing what the syllabus states [Vinner, 1997]. In order to function as an $\mathbf{S_1}$, the teacher must not know that he is there to promote this passing without knowing (pseudo knowledge, Vinner). Students know many things, especially how to pass. However, what they know still better is that the teacher wants the game to go on and things to work well. They also know that the teacher does not know that this is what he wants. In order to be there, the teacher has to inebriate himself with his phantasm of minister of knowledge.

The clearest possible example of such a knowledge is provided by the people and the king, so brave and courageous in his cherished portrait although everybody knows that he is fat and flaccid and that he mounts a wooden horse. Everybody helps him to hide his feet of clay. The students are not interested in exhibiting the master's ignorance either. At most they threaten to reveal it to those who understand easily, in order to sit beside him and help him fill in the marks in the grade sheet or to get a recommendation for a fellowship. Such a truth that can never be fully stated as a truth determines the gap of the big-Other. The signifier of the castrated subject is $\not{S}$. Lacan symbolizes such a relation between $\mathbf{S_1}$ and $\not{S}$ by $\frac{S_1}{\not{S}}$, the bar indicating that $\mathbf{S_1}$ is supported by and hides S.

But it is not because of a personal capacity like physical force or ability with the sword that the king makes himself obeyed. It is the serfs who make the king what he is by the exercise of obedience to the symbolic law. Lacan symbolizes the basic relation between $\mathbf{S_1}$ and $\mathbf{S_2}$ as $S_1 \rightarrow S_2$. The arrow defines the impossible: "(...) il est en effet impossible qu'il y ait un maître qui fasse marcher son monde. Faire travailler les gens est encore plus fatigante que de travailler soi-même, si l'on devait faire vraiment. Le maître ne le fait jamais" [Lacan, 1991, p. 202].

What the student expects at the beginning of the course is to be able to identify loss of *jouissance* with social promotion. The effort should be paid off by passing. We shall explain this. The student needs a thick copybook or binder, full of solved exercises, all equal to the first sample; simple applications of the same formula. This copybook, full of blue marks that s/he shows to her/his proud parents should lead him/her to promotion in the school credit system at the end of the year. It should never be revealed that the credit is the *cause* (double meaning) of his/her desire, the biggest kick that he hides and that constitutes him/her as a social agent - a student. The signifier of the cause of desire is the small-a, **a.** At the end of the year, like the slave at the end of history in Hegel, the student reaches a sort of absolute knowledge and finds out that the cherished copybook fits better in the bonfire of useless and meaningless statements. However, next year, touched off by a kind of compulsive bias for repetition, s/he starts all over again. According to Lacan "la répétition est fondée sur un retour à la jouissance (...) dans la répétition même, il y a déperdition de jouissance" [Lacan, 1991, p. 51].

We have described the discourse present at the first moment of the course, when the teacher introduces the *work contract* or leaves it implicit. The work contract defines the conditions for obtaining credit in the course: how many exams, what textbook, how grades will be determined, etc. This discourse may be extended well beyond the inaugural moment and may constitute the overwhelming classroom discourse during the year. Then we will have a prototype of a certain traditional teaching. The teacher solves one exercise on the blackboard and assigns similar ones to the students for drilling. Students should sit upright in matrix position, look at the blackboard, pay attention and be silent. From the teacher we hear: - *This is the way to do it. That is what I want. Do it once more.* Many textbooks are still organized to facilitate this kind of teaching.

The student gives up learning in order to keep the school's game going. Learning would imply passing to another knowledge, distinct from knowing how to pass. The student would cease being who s/he is to become somebody else - a learner. Most of them refuse to face this kind of death. The knowledge written in the thick copybook is his/her lost jouissance. S/he produced it but the *jouissance* was lost in the repetitive actions. The slave needs the master precisely in order to hide from himself that he has exchanged freedom for life and that he loses *jouissance* in producing for somebody else. What the slave produces is not his own; but without producing it would be impossible for him to maintain the game of life. The relation between **S_2** and **a** is represented $\frac{S_2}{a}$, the bar having the same function as before. It means that knowledge **S_2** is supported by and is an alibi for the credit system **a**. Of course, the castrated master cannot be put face-to-face with the credit system. If this happens, the whole plot is revealed and collapses. Between $\$$ and **a** there is a barrier that Lacan denotes by a black triangle $\$ \blacktriangle a$.

"(...) il n'y a pas de rapport entre ce qui va plus ou moins devenir cause du désir d'un type comme le maître qui, comme d'habitude ne sai rien, et ce qui constitue sa vérité (...) La barrière (...) c'est la jouissance (...) en tant qu'elle este interdite, interdite dans son fond (...) pas besoin de reagiter les phantasmes mortiphères" [Lacan, 1991, p.124].

Putting together the signifiers and their relations that we have described up to this point, we get what Lacan calls a four-legged diagram representing the four discourses. The first one is the *master's discourse*.

$$\frac{\text{master-signfier}}{\text{subject}} \underset{\blacktriangle}{\rightarrow} \frac{\text{knowledge}}{\text{jouissance}} \qquad\qquad \frac{S_1}{\$} \underset{\blacktriangle}{\rightarrow} \frac{S_2}{a}$$

There are four functional positions in the four-legged diagram. Those functions are maintained insofar as the signifiers **S_1**, **S_2**, **a**, $\$$ circulate to produce the four discourses. The master's discourse is the one from which the other three are obtained. Positions and functions will be maintained while the signifiers are shifted through the diagram in order to obtain the three other discourses: the university's, the hysteric's and the object's discourse.

The upper-left position is the position of the *agent*, conceived not as the one who acts, but as the one who is put into action by the demand. In the master's discourse, this position is occupied by the **S_1**, master or teacher. It defines the function of *desire* insofar as it stirs the Other's desire. In each of the four discourses, the signifiers will inherit something from the positions they occupy and will leave a trace of their presence in this position. The **S_2**, the slave or the student, is in the upper-right position which is the position of *work*. From this position the demand is exerted. It also defines the functions of the *big-Other*, such as providing meaning to what is said, maintaining the law, and the ideology, etc. The $\$$, the castrated master or the ignorant teacher, is in the lower-left position, the position of *truth*. It also characterizes the function of *truth*. Finally, the **a**, the slave's production or the credit system, is in the lower-right position, the position of *production* that characterizes the function of *loss*.

We shall not have space to explain the other three types of discourse. We can only present their diagrams and leave their application to classroom situations to the reader's imagination.

THE MASTER'S DISCOURSE
It is impossible to rule

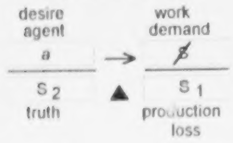
$$\begin{array}{ccc} \text{desire} & & \text{work} \\ \text{agent} & & \text{demand} \\ \frac{S_1}{\$} & \underset{\blacktriangle}{\rightarrow} & \frac{S_2}{a} \\ \text{truth} & & \text{production} \\ & & \text{loss} \end{array}$$

THE UNIVERSITY'S DISCOURSE
It is impossible to educate

$$\begin{array}{ccc} \text{desire} & & \text{work} \\ \text{agent} & & \text{demand} \\ \frac{S_2}{S_1} & \underset{\blacktriangle}{\rightarrow} & \frac{a}{\$} \\ \text{truth} & & \text{production} \\ & & \text{loss} \end{array}$$

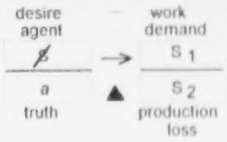
THE OBJECT'S DISCOURSE

It is impossible to analyze



THE HYSTERIC'S DISCOURSE

It is impossible to make desire



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Which is the Shape of an Ellipse?

A Cognitive Analysis of an Historical Debate

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ABSTRACT This report starts from the cognitive analysis of an imaginary debate - reconstructed with excerpts from historical sources - concerning the shapes of particular sections of a right cone and of a right cylinder. The analysis, based on the theory of figural concepts (Fischbein, 1993), suggests the following hypothesis. When conic sections are concerned, a break between the figural and the conceptual aspects is expected and is not easy to be overcome. An exploratory study with expert university students was carried out to validate the hypothesis exploring what kind of conceptual control, if any, were students able to mobilise in order to overcome the break. After reporting the findings of the study, we suggest which tools of semiotic mediation (Vygotsky, 1978) could be introduced to make the students acquire the possibly lacking conceptual control.

1. The Task

An imaginary debate has been constructed with excerpts of historical sources from different ages. It concerns the shape of a particular conic section. Read it carefully.

Serenus: *Since I know that many expert geometers think that the transversal section of the cylinder is different from the one of the cone that is called ellipse, I have thought that they must not be allowed to make such a mistake. Actually it is absurd that geometers speak about a geometrical problem without giving proofs and are attracted by truth appearances, versus the spirit of geometry. However, since they are convinced of that and I am convinced of the contrary, I shall proof 'more geometrico' that both solids have a section of the same kind, rather identical, provided that the cone and the cylinder are cut in a suitable way* (The Sections of Cylinder and the Sections of Cone, IV century A. D.)

Witelo: *All the ellipses that are sections of the acute-angled cone are larger in the side close to the base of the cone, this is not true for the ones obtained as sections of the cylinder. It happens because of the sharpness of the cone and the regularity of the cylinder. In fact, on the one side, let us consider the intersection of the axis of the cone with a line perpendicular to a side of the axial triangle; if we draw a circle on the cone with that centre and we imagine a cylinder with this circle as the basis: it is evident that the bottom piece of the cone is external to the cylinder whilst the top piece is internal. Hence the bottom part of the conic section contains the bottom part of the cylindrical section, whilst the top part of the cylindrical section contains the top part of the conic section. On the other side, the two parts of the cylindrical section are equal because of the regularity of the solid and of the equality of the angles with the axis. Hence the thesis follows* (About Perspective, about 1200 A. D.).

Dürer: *I do not know the German names of the (conic) sections, but I suggest to name the ellipse, egg-shaped curve, as it is identical to an egg* (Treatise on measuring by rule and compasses on the line, on the plane and on every body, 1525).

Guldin: *It is necessary to avoid the mistake of those who think that the (conic) ellipse is narrower in the part close to the vertex of the cone and larger in the part close to the basis of the cone: on the contrary they are very similar* (Centrobaryca 1640).

What do you think? How could you convince an interlocutor (e. g a school fellow of yours) whose opinion is different from yours? And how could you convince a Junior High School student?

2. A Cognitive Analysis of the Historical Debate

The above imaginary dialogue reconstructs some elements of an historical debate between voices from two antagonist and complementary worlds: the one of theoretical geometry and the one of practical geometry (Balacheff 1997).

Theoretical geometry. Since the age of Apollonius a deep understanding of the properties of conic sections had been gained. However, most of the properties were expressed through relationships, immediately related to neither the shape of the cone to be cut nor to the shape of the section. For instance, the theory of Apollonius was based on proportions and applications of areas, by means of which the very 'symptoms' (i. e. characteristic properties) of conics were expressed in the secant plane, despite the initial 3-D approach. This process based on calculations culminated in the algebraic representation of conics by equations in the 17th century. A simple and meaningful link between the 3-D approach and the 2-D approach to conics was looked for by mathematicians for centuries until Dandelin (1822) succeeded in relating the focal properties of a conic to a configuration with a conic section and two spheres tangent to the cone and to the secant plane, where the points of tangency are the very foci (see fig. 6; for an elementary proof see Hilbert & Cohn Vossen, 1932; see also http://155.185.1.61:80/labmat/dandin.htm).

Practical geometry. Conic sections were studied also with the purpose of applications, such as setting sundials, constructing burning mirrors and drawing in perspective. Bacause of the modelling process the 3-D generation of conics was focused. Witelo and Dürer belonged to this tradition. In particular, Dürer applied the graphic method of double projection, practiced in the painter workshops to conic sections. However, in spite of the keen (and right) method, that was reconsidered in the late 18th century by Monge, he drew an egg-shaped curve (see the fig. 7 from Dürer, 1525), instead of an ellipse, probably deceived by arguments similar to the ones expressed by Witelo. Guldin witnessed the permanence of the misconception more than one century later.

The above outline (for more details see Bartolini Bussi, to appear) shows the existence of two relatively independent worlds, that came in contact with and nourished each other from time to time. However the relative independence from each other created the conditions for the birth of autonomous manners of viewing and styles of reasoning.

The break between the arguments used in theoretical geometry and in practical geometry seems interesting to be investigated from a cognitive perspective, besides the historical point of view. Let us try to analyse it, in this specific case, according to the theory of *figural concepts* (Fischbein, 1993).

The geometrical concept of *cross section* (i. e. the figure obtained as the intersection between a plane and a surface or possibly the solid bordered by it) proves to be difficult in general (Mariotti, 1996), since it is necessary to forget the global appearance of the solid and infer from 'outside' what happens 'inside'. For instance, think of a cube cut by a plane oblique in respect to all of its edges (see the fig. 1). It is hard to overcome the possible conflicts between the figural aspect, deeply affected by the global shape of the object and its components (all the faces of the cube are squares) and the conceptual aspect, that concerns the properties of the intersection between the set of the points of the cube and the set of the points of the given plane. On the one side, there are many implicit properties of the solid, which become determinant in order to characterise the shape of the cross section, but cannot be immediately grasped, or which conflict with perceptual attributes; on the other side the shape of the solid can hide other properties. For instance, the symmetry of a section is hardly recognised since symmetry (unlike other geometric properties) is neither invariant for intersection nor easily related to the properties of the two sets to be intersected.

The above analysis shows that the solution of a section problem implies a high conceptual control, which is supposed to be more problematic in practical geometry, where a supremacy of the figural aspect appears and an adequate conceptual control is not always available. Drawing on this analysis originated from an historical case, we guessed that even experts might not be able to control practical arguments based on the supremacy of the figural aspects, because of the break existing between figural aspects and the available conceptual instruments, with a resulting conflict between the strategies applied in the two cases.

3. The Exploratory Study

3. 1. PURPOSE. We made the following cognitive hypothesis: *When conic sections are concerned, a break between the figural and the conceptual aspects is expected and is not easy to be overcome (even for students formally educated).* Our investigation aimed to explore what kind of conceptual control, if any, were students able to mobilise in order to overcome it. We wished to observe how they succeeded in defending a position under the pressure of an imaginary debate, in order to by-pass the break and harmonise the figural and the conceptual aspects. If the above hypothesis had been validated, a new didactic problem should have arisen: *how could have the teacher made the students acquire the (possibly) lacking conceptual control.*

3. 2. METHOD. 14 students from the courses of Elementary Mathematics from a Higher Standpoint (i. e. a Course on Epistemology of Mathematics for prospective teachers), taught by the two authors in the 3rd - 4th years of the courses of Mathematics, accepted to take part in an afternoon problem solving session in their respective Universities (A and B). Both groups had taken two one year-courses - i. e. nearly 300 hours - in geometry including linear algebra, vectorial, euclidean, affine and projective spaces and the algebraic study of conics and quadrics. In these courses they had information about the intersections of a cone and a cylinder with a plane. Hence they could be considered experts: there was no doubt about the fact that in both cases it is possible to obtain an ellipse. They were divided into small groups (2 trios - A2, B5 - and 4 pairs - A1,B1+B2,B3,B4), given individual copies of the text of the § 1 and asked to produce only one answer for each group (in one case, a pair did not succeed in reaching an agreement, and the group produced two texts - B1, B2).

3. 3. DATA. The problem solving session lasted two hours. We collected 7 written protocols (A1 A2; B1 B2 B3 B4 B5) with drawings and paper models. The groups realised various and different explorations. We shall give only some details.

3. 3. 1. Witelo's and Dürer's Arguments. Some groups guessed the reasons for the mistake and tried to find the bug . We shall focus on three different issues:

The symmetry reasoning. Some drawings (e. g. fig. 3) and the transfer of a property of symmetry from the cone to the section might suggest the idea that the centre of the conic section (if any) is the intersection of the axis of the cone with the secant plane. A group (A1), tried to find a symmetry between the two parts of the section that are on the different sides of the triangle of the paper model (fig. 2). When the model failed, they drew a right cone and a right cylinder with the same base and a secant plane and tried to show that the two sections are concentric ellipses, by estimating their *'distance'*. But when they imagined to incline the secant plane more, they saw that *the 'distances' change in a different way: on the one side the curves becomes closer and on the other farther.* This was even more puzzling because it stressed the difference between the cylindrical and the conic sections against their wish of eliminating differences. However it convinced them that the ellipses might be not concentric, i. e. that the centre of symmetry (if any) might be out of the axis of the cone.

The containing-contained reasoning. A group of students (B3) drew a right cone and a right cylinder with the same axis as described in Witelo's argument. Then they drew the sections with the same plane. They obtained two closed curves intersecting in two points (fig. 4). The particular configuration might explain, in their opinion, why there is no contradiction in having a *'true' ellipse* for the conic section that is *contained* in the cylindrical section on the top side and that *contains* the cylindrical section on the bottom side.

The limit case reasoning. A student (B2) explained Dürer's mistake by means of a limit argument. He drew the section of a regular pyramid and a regular prism with hexagonal basis by means of orthogonal projections and commented: *It is evident that the two sections are not of the same kind, in fact, whilst the prism section is symmetrical, this is not true for the pyramid section. We can imagine that this disproportion is maintained when, by increasing the number of sides, the pyramid and the cone approach at the cone and the cylinder. [...] [In the case of the cone] perception might suggest a not symmetrical curve (i. e. Dürer's egg). The mistake is reinforced by the common pointwise construction of sections, that is usually made by choosing on the circle the vertices of a regular polygon.* However the student did not try to explain the apparent conflict between the symmetry of the section of the cone and the asymmetry of the section of the pyramid.

3. 3. 2. Models. The students referred to both 3-D models and 2-D models (drawings)

3-D models. In one case (A1), a pair of students started to cut, fold and glue paper and to make some models. They were very hopeful because *a good paper model could be useful to convince Witelo practically, as Witelo is speaking in practice and not in geometry. He's not giving any proof.* The most promising model, that was handled for a lot of time, consisted of two equal isosceles triangles stuck together orthogonally along their heights (fig. 2). The two students tried to describe with hands or pencil the section in this model but without success. They hoped to find some symmetry between the two parts of the section that are on the different sides of one triangle but they did not succeed. This practical experience helped them later to exit from a blind alley (see § 3.3.1). Two groups of students (B3 and B5) suggested to use (not available) concrete wood models to explain the result to young pupils too.

Drawings. From the very beginning, while reading the given text, all the students started drawing. Since the problem concerns a 3-D configuration, different kind of drawings were produced, according to their previous experience in high school; however in most cases the scarce - if any - mastery of effective drawing abilities did not allow them to find conclusive evidences. For instance, a group (A2) focused attention on only one position of the secant plane from the very beginning (fig. 3): the little inclination of the plane suggested a false symmetry (see § 3.3.1); hence, for the whole session, they tried to prove a false conjecture without success.

3. 3. 3. Proofs. Only one student (B2) introduced proofs in his answer.

Synthetic. He reminded a way to relate the focal property of the ellipse as a locus of points to a conic section, by drawing Dandelin's configuration (fig. 6). He told to have seen a model once and to have been struck by the ingenuity of the method.

Analytic. No student tried to approach the problem by means of analytic geometry, even if some groups turned to equations later. Yet only the student B2 succeeded in proving that *cutting a cone with a suitable plane, the conic section is an ellipse*, by a fusion of synthetic and analytic instruments: the synthetic part allowed him to avoid the problematic recourse to the system of the equations (in three variables) of the cone and of the plane by shifting the reasoning to the secant plane only, where a local system of cartesian coordinates was introduced; in this system the equation of the section was calculated correctly, using synthetic geometry, and reduced to the known canonical form (fig. 5). But he was not satisfied by this proof *'more analitico'* (as he called it), *because it cannot tell anything to a person - like Witelo - who does not know ellipses as equations.*

3. Discussion

Although its small size, the exploratory study confirms our hypothesis. Data highlight the break between the figural and the conceptual aspects in the field of conic sections and the great difficulties met to overcome it. The university students had information about conic sections and nobody doubted about the truth of the known statement. Yet only one student (B2) succeeded in proving that the conic section under scrutiny is an ellipse. All the students looked for a direct argument against Witelo, that is an argument that would not break the link with the figural aspect. The group B3 contrasted Witelo's argument based on set inclusion without offering a complete proof. A lot of students expressed their faith in (not available) concrete models. Unfortunately it would not be possible to control Witelo's argument only by refining perception: the ellipses are not 'seen' as ellipses (as it is shown in his beautiful analysis of the ancient drawings of circular wheels and elliptical shields, depicted according to Euclid's Optics, by Knorr, 1992) and direct measuring with comparison of segments might be impossible on the wood models proposed by the students. Actually a direct argument to exert a conceptual control on Witelo's figural argument does not exist.

One could defend the elliptic form of a particular conic section in different ways, by producing rigorous proofs: e. g. in the style of Apollonius, by proving that the section has two orthogonal conjugate diameters (i. e. axes of symmetry); in the style of Serenus, by constructing (and proving the construction) a cylinder with the same section of a given cone; in the style of post-cartesian geometers by using analytic geometry (Herz-Fischler, 1990). All the students had been trained in analytic geometry, yet they (with one exception) did not use equations effectively. The cause could be the failure in managing 3-D analytic geometry, but also the voluntary choice of excluding a not appropriate way of contrasting Witelo's argument. Actually, even if the task did not ask it explicitly, the students seemed to enter by themselves into a *voices-echoes game* (Boero & al., 1977), where the two voices represented practical geometry and theoretical geometry. They tried to express a *dissonance* by appropriating Witelo's voice and refining it without success. Even the only student (B2) who built a synthetic-analytic proof was not satisfied, because he thought that it could not have convinced Witelo. Actually in the sequence of algebraic manipulations of a formula it might be impossible to keep track of a sequence of geometric steps to which the formal manipulations should correspond.

From these data, can we conclude that no form of conceptual control of Witelo's figural argument is available for these university students? We do not think so. At least

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two purely synthetic arguments do exist that can be expressed in elementary terms and can be explored through perception, up to becoming tools of geometrical reasoning: (1) the (revised) Dürer's method of double projection (fig. 7); (2) the Dandelin's theorem (fig. 6). The former may be improved by the recourse to dynamic softwares (like Cabri or Geometer's Sketchpad) which transform the pointwise construction into a 'continuous' one and make 'visible' what happens in the secant plane while we shift from the bottom to the top or viceversa, explaining also why pyramids and cones have different properties. The latter creates a visible link between the conic section and its foci. Yet, two problems arise. First, they do not address directly the symmetry of the section, that could only be undirectly inferred. Second, to restore the harmony between the figural and the conceptual aspects, a deep transformation of the original figure is required: in the Dürer's method, the object is broken into three different views (on the vertical, the horizontal and the section planes) and, in the Dandelin's theorem, two auxiliary spheres are introduced into an already complex configuration. Both appear as 'tricks' that can be appropriated (but cannot be discovered without help) by students. We have here two very examples of tools of *semiotic mediation* (Vygotsky 1978), which may be introduced from the outside (by the teacher, by a book, by a model, by a guided practice) and may have the effect of controlling the immediate reaction based on perceptual appearance. Further studies are needed, but we guess that the very analysis of the quality and of the amount of help necessary to restore the broken harmony between the figural and the conceptual aspects could give a useful pointer of the relative difficulty also in the case of advanced geometry tasks for the tertiary level, such as the one we have presented in this report.

Acknowledgements. We wish to thank Marcello Pergola who first called our attention to this beautiful piece of the history of conic sections.

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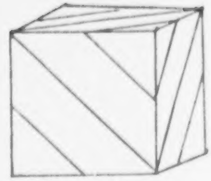

fig. 1 ('slicing' a cube)

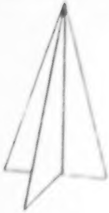

fig. 2 (paper model - A1)

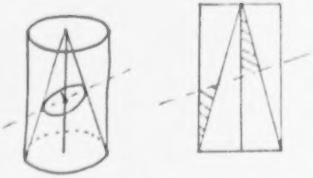

fig. 3 (A2)

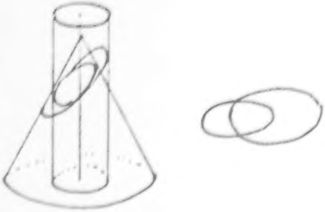

fig. 4 (B3)

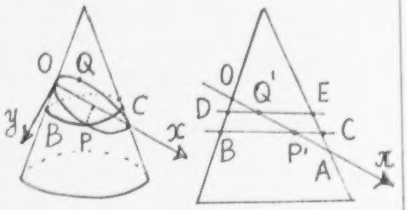

fig. 5 (B2)

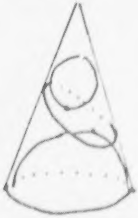

fig. 6 (B2)

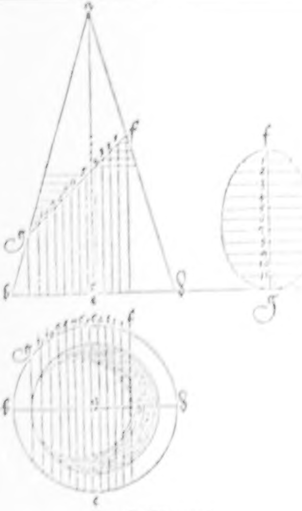

fig. 7 (Dürer)

The students' free hand drawings on squared paper have been retraced on white paper for the purpose of reproduction.

CHILDREN'S UNDERSTANDING OF THE DECIMAL NUMBERS THROUGH THE USE OF THE RULER

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Abstract:*This is an exploratory study about the use of ruler to introduce the concept of decimal number, in the normal classroom curriculum, with third-grade children. We propose that the children's use of the ruler can have a mediational role in their understanding of the additive structure underlying the standard written decimal notation. In order to achieve our objective, we have designed a classroom practice that engages students in a sustained mathematical activity which requires an extensive use of the ruler to accomplish different functions (measuring, drawing segments, ordering and approximating decimal numbers). Opportunities and constraints in children's use of the ruler to achieve the educational goal are presented.*

Framework

Educational models that are developed from the sociocultural and the constructivist perspectives (Cobb, 1994; Confrey, 1995) have criticized the formalistic models of teaching/learning mathematics in elementary classrooms, and have proposed new directions for implementing learning environments.

Effective learning does not consist in the acquisition of a fixed amount of knowledge, transmitted by a teacher, but mathematical thinking develops in rich mathematical environments, in which children have many opportunities to deal with dilemmas and problems, make use of tools, share ways of doing things, utilize previous knowledge in order to contribute to an intended goal in practice.

Saxe (1991) interprets the development of mathematical thinking in terms of shifting relations between form and function. Cultural forms such as the ruler and the decimal symbol convention are acquired and used by individuals to accomplish different cognitive functions (like measuring, calculating, ordering) that emerge in a cultural practice. Children appropriate and specialize prior forms to accomplish new cognitive functions. In order to support children's use of cultural forms as a means to achieve relevant educational goals, the negotiation of a certain amount of shared meanings is required. Therefore, classroom discourse provides resources for, challenges to and constraints on children's thinking (Forman, & McPhail, 1993). Effective learning environments support the participants in their moving from 'primitive doing' of mathematical actions through the use of tools, towards progressively sophisticated abstractions (Pirie, & Kieren, 1992). In such environments, the learners do not construct their representations of mathematical symbols in a vacuum, but have manifold opportunities to ground their construction of mathematical meanings in 'situation-specific imagery', as elaborated in practice through the use of cultural forms (McClain, & Cobb, 1996).

The formal approach in introducing the decimal numbers in elementary classrooms

In previous studies (Bonotto, 1993; 1996), 10-11 year italian children's conceptual obstacles in ordering decimal numbers are analyzed. The findings are consistent with classical researches (Nesher, & Peled, 1986; Resnick, et al., 1989).

It was hypothesized that such findings may depend not only on the inherent difficulties of the subject matter but also on the teachers' conceptions and educational strategies. Many teachers introduce the decimal numbers by extending the place value convention; they tend to spend little time to let the children understand the meaning of the decimal number symbols and reflect on the decimal number properties and relationships; efforts to connect decimals and decimal measures lack. As consequence, children do learn to carry out the required computations, but they have difficulties in mastering the meaning of decimals, the relationship between meanings and written conventions and between fractional representation and decimal representation, and finally in ordering sequences of decimals.

Measuring activities as an alternative introduction to the decimal numbers

According to innovative instructional approaches, we mantain that children's decimal number understanding can be fostered in rich classroom environments, where learners can transfer their out-of-school knowledge and utilize familiar tools (such as the ruler) to accomplish a recurrent set of mathematical activities, and where they can share some minimal presuppositions about the problem definitions and the goals.

We propose that a set of measuring activities that require an extensive use of the ruler can offer the children good opportunities to move toward the construction of an encompassing numerical structure, which integrates in a consistent whole both the natural and the decimal number systems.

The ruler is a cultural artifact (Saxe, 1991) which can offer the children a first approach to the decimal number as the result of a given measurement. On the ruler, 'mathematical facts' are represented through its signs: the natural number sequence is visible, and fractionary parts are marked. Therefore, the ruler can offer a 'situation-specific imagery' of the additive structure of the written decimal number notation, which supports the children's progressive understanding. For example in order to draw a 3.15 dm segment, the child firstly draws a 3dm line and marks the final extreme, then she/he adds a 1cm line to it, and finally a 5mm line, and expresses each affixion as 'plus', or 'and'. The child can understand that if there are two decimal digits after the decimal point, then there are units, plus tenths plus hundredths, and that each digit specifies how many parts of a given magnitude are included in the addition. The learner is expected to form images out of her/his actions through the use of the ruler, and to visualize relevant properties. The child can map this visualization onto the decimal number representation to attribute a meaning to the decimal digits after the decimal point; her/his ability to solve ordering problems is enhanced.

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Furthermore, the activity of measurement can offer a common reference in communication, by making mutually explicit one's own operations and understandings.

The research:

We have designed a classroom practice that engages students in sustained mathematical activities which require an extensive use of the ruler to accomplish different functions.

In the first year of experience (Basso, Bonotto, Sorzio, 1996), we had asked third graders to draw, measure, compare segments expressed in centimeters, because it is the unit of measure the children are familiar with in their classroom activities. However, the introduction of the second decimal digit (expressing tenths of millimeters) had been too difficult to understand for many children because it is not marked on the ruler.

Therefore, we devised a second experience in which other 8-9 year children were asked to represent given decimal numbers as segments, to measure and represent given segments in the written symbol notation, to compare, and to order decimal numbers. All the decimal numbers were represented in decimeters to make the second decimal digit representable by a mark on the ruler.

In order to make the decimeter a meaningful unit of measure, we introduced the tasks as representing some problematic situations in the imaginary world of an Olimpic Long Jump Game in Lilliput.

Research objectives: data are gathered and analyzed about:

-the children' understanding of the signs on the ruler;

-the children's use of the ruler to measure segments, and to draw segments of required lenght;

-the children's understanding of measuring as a process of approximation;

-the children's understanding of the additive structure underlying the standard written decimal notation;

-the children's process of detachment from the representation on the ruler and from the presence of a given unit of measure;

-the children's understanding of the density of the enriched number line with decimals.

The mediating role of small group and classroom discurse in enabling the children to achieve the educational goal of connecting the representation on the ruler and the standard written notation is also analyzed.

Subjects: 15 third grade children (aged 8-9 years) in a small school in a village (NE Italy) participated to the experience.

We observed the children as they played the game once a week, for a period of six weeks, two hours per session. Data were gathered from videorecordings, participant observations and children's protocols.

In the first hour of each session, the children were sorted in small groups of four learners each to solve the tasks; in the second hour, the children discussed their reasoning processes and the results in a classroom discussion led by the teacher. *Material:* Each child was given a 3 decimeter ruler, a set of papers showing two partially filled in tables recording the numerical representations of jumps, and a series of segments representing jumps to be measured.

Procedure:: 1. In order to introduce the first decimal digit, the children were given the first partially filled in table representing jumps in the decimal notation, and a set of segments representing jumps to be measured. The children were asked to to represent the measures at the first decimal digit in the table;

2. eventually (in the third session), the children were given the second partially filled in table in which the second decimal digit representing the millimeters was introduced by Lilliputians through the use of a more precise measuring tool; they were asked to measure and represent jumps in the table;

3. finally, the children were asked to make comparison and ordering inferences about decimal numbers and to mentally check whether there are numbers between two given numbers. Children were expected to perform these tasks without relying on the ruler, in order to begin a reflection about the number line properties, e.g. density.

Cognitive functions accomplished by the children: measuring given jumps; drawing jumps of required lenght; converting measures in decimal numbers to fill the table; evaluating each Lilliputian's best jump; ordering the best jumps to ascertain the winner, the second, the third in the game. Comparing the winner's jump and the Lilliput world record, and evaluating the difference.

Ordering measures expressed in the decimal number notation; evaluating which of two measures better approximates a third measure.

Ordering, comparing, and approximating decimal numbers.

Discussion

We briefly present some excerpts in which the individual constructions of the decimal number understanding are mediated by the use of the ruler and by small group and whole classroom discussions.

In the following passage from the third session, Marta's difficulty in understanding the written decimal notation is highlighted. She is comparing two lilliputians' jumps (Alberto's 3.20 dm and Carlo's 3.28 dm).

Marta (trying to draw the 3.20 segment as if it were 3 dm e 20 cm.): "*No space... It's 20, I am drawing the 3.20 dm, and I can't get the 20cm being in* (my drawing sheet)"

researcher: "*the 20... what do they represent?*"

Marta: "*they are 20 millimeters, then. If I had to come to 20* (she means centimeters), *I'd come to here* (outside the leaf). *Therefore, they are 20 millimeters*!"

Michele (he already drew the 3.28dm segment): "*have you seen? these are 3 dm, these the 2cm, and these the 8 mm* (for each additional segment he points to the corresponding mark he signed on the 3.28dm segment)"

Marta: "*where are the 2 centimeters? Aha*!"

At the beginning, Marta is not able to have an image of relevant properties of the given decimal number, and she interprets 3.20 as it were a juxtaposition of whole numbers, the decimal part representing how many centimeters (20) should be added to the decimeters represented on the left of the decimal point. The measuring activity offers a reference for Marta to have a specific imagery of her intended action, that enables her to understand that each decimal digit position represents the magnitude, and its value represents how many parts of a given decimal magnitude there are.

Comparison: during a small group discussion (fourth lesson), some children are ascertaining the smaller between 8.1 e 8.15

researcher: "*which number is smaller: 8.1 or 8.15?*"

Thomas: "*8.1 is larger*"

researcher: "*why do you think so?*"

Thomas: "*because here* (8.15) *there are millimeters*"

researcher: "*and therefore you think it is smaller, don't you?*"

Thomas is puzzled, others disagree.

Moreno: "*8.1 e 8.15... it would be 8 dm and 1 cm., and 8.15* (would be) *8 dm, 1 cm e 5 mm* (the other children agree). *This* (8.1) *had only 1 centimeter, and this* (8.15) *was 5 millimeters ahead.*"

Thomas correctly understands millimeters as connoting smaller parts than centimeters, but he incorrectly concludes that the total value of the term with more digits in the decimal part must be smaller. Moreno interprets the given decimal numbers as measures, and therefore he understands each of them as representing a summation; he compares the two numbers in terms of lenghts.

In this classroom discussion (fourth lesson) the process of representing a given measure in the standard decimal notation is highlighted:

Each child is presented a 3.25 dm drawn segment and asked to represent it in the standard decimal number notation.

Marta (measuring the segment): "*3 dm, 2, let me count the millimeters ... 5.*" she writes "3. 2.5".

Teacher: "*can you mark it* (select a unit of measure)? *You wrote '3.2.5'* (writes it on the blackboard)"

Marta: "*32.5* (puzzled)"

The teacher points to the non standard notation '3.2.5' , and asks whether the given segment can be expressed in a different notation: Marta repeats "*32.5*"

teacher: "*you may be right, but which mark is correct?*"

(.....)

Mattia: "*32 cm point 5 mm*"

teacher: "*can you write it differently?*"

Mattia "*3.25*"

The teacher (writing 3.25): "*how do you mark it ?*"

Mattia: "(...) *decimeters* "

teacher: "*why do you think so ?*"

Mattia: "*because I think a Lilliputian cannot jump 3 centimeters and 25 millimeters, he can jump 3 decimeters and 25 centimeters* "

researcher: "*no, 3 decimeter, 2 centimeters and half, not 25 centimeters. (....) how much 25 cm is ?*"

Mattia: "*2 decimeters e 5 centimeters*"

Marta chooses a numerical representation that reflects her additive action of measuring; when asked to translate in the correct form, she relies on the numbers she reads on the ruler (representing centimeters). Mattia's answer reflects his conflict between his experience with measures expressed in decimeters and the numbers on the ruler which represent centimeters; however the ruler offers a concrete representation that enables him to understand his mistake.

In the fifth session, the teacher expects the children to detach their understandings about the decimal numbers from the representation on the ruler and from a given unit of measure.

teacher (writing 3.15 on the blackboard): "*How can you read it?* "

The whole class "*3 point 15*"

teacher: "*make your observations, Mattia would you come to the blackboard*"

Mattia: "*3.15 you can say 3 point 1 centimeter point 5 millimeters*"

teacher: "*what is the 3?*"

Mattia: "*decimeters, if the unit of measure is decimeters*"

teacher: "*if there were no unit of measure?*"

Battista: "*centimeters are ten times smaller, the millimeters one hundred ...*"

Elisa: "*3 is the unit*"

teacher: "*3 what?*"

Mattia: "*unit*"

teacher: "*as to say... what?*"

Mattia: "*3 decimeters*" (...)

teacher: "*why? here there is the decimal point, what is this right there?*"

Child: "*the smaller pieces*"

teacher "*smaller than what?*"

Silvia: "*than 3*"

Mattia: "*no, because 5 is larger than 3* (maybe he is misinterpreting Silvia's utterance)

Giulia: "*no, because 3 is decimeters,* (that is) *30 centimeter unities*"

(....)

teacher: "(how do you represent it) *in terms of just numbers?*"

Child: "*unit of a number*"

teacher: "*what is the 1?*"

Child: "*another unit of number*"

Battista: "*ten times smaller*"

teacher: "*than what?*"

Battista: "*than 3*"

teacher: "*what is 5?*" (....)

Battista: "*another unit of number, (which is) one hundred times smaller*"

teacher: "*why do you think it is one hundred times smaller?*"

child: "*because there are the millimeters*"

Mattia: "*and afterwards there would be the thousands ...*"

teacher: "*where would you place the parts (that are) smaller than 5?*"

Mattia: "*right here* "(he correctly points at the place after the last decimal digit)

teacher: "*what would you place there, if nothing is signed?*"

Mattia: "*a 0*".

Although the teacher is trying to lead the children towards a more abstract thinking about the decimal number meaning, she is leaving underspecificated the new frame of mathematical discourse her utterances imply. The children are utilizing their prior experience with the ruler as a means to progressively gain more indications about the teacher's intentions and to make the discussion go on.

To calibrate their understanding, they are utilizing their additive conception of decimals, and are interpreting the place value convention in terms of 'smaller pieces' or 'remains'. However, their difficulty in moving towards a conception of the pure number persists, since the conflict they still experience between the meaning to be attributed to each decimal digit value -which represents how many parts of a given magnitude there are- and the meaning to be attributed to each decimal digit position -which represents its magnitude.

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CONSTRUCTION OF MULTIPLICATIVE ABSTRACT SCHEMA FOR DECIMAL-NUMBER NUMERATION

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This paper reports on an intervention study planned to help Year 6 students construct the multiplicative structure underlying decimal-number numeration. Three types of intervention were designed from a numeration model developed from a large study of 173 Year 6 students' decimal-number knowledge. The study found that students could acquire multiplicative structure as an abstract schema if instruction took account of prior knowledge as informed by the model.

Baturo (1997) explored students' acquisition of, and access to, the cognitions required to function competently with decimal numbers. One hundred and seventy-three Year 6 students from two schools (different socioeconomic backgrounds) were tested with a pencil-and-paper instrument that included items designed to assess number identification, place value, counting, regrouping, comparing, ordering, approximating and estimating for tenths and hundredths. As a result of analyses of the students' performances and of the cognitions embedded in decimal-number numeration processes, Baturo developed the numeration model shown in Figure 1 to show these cognitions and how they may be connected.

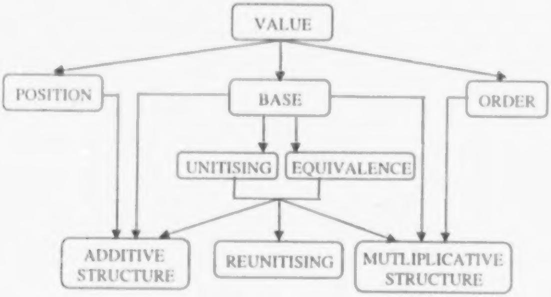

Figure 1. Cognitions and their connections embedded in the decimal number system (Baturo, 1997).

The model depicts decimal-number numeration as having three levels of knowledge that are hierarchical in nature and therefore represent a sequence of cognitive complexity. *Level 1 knowledge* is the baseline knowledge associated with *position, base* and *order*, without which students cannot function with understanding in numeration tasks. Baseline knowledge is unary in nature comprising static memory-objects (Derry, 1996) from which all decimal-number numeration knowledge is derived. *Level 2* *knowledge* is the "linking" knowledge associated with *unitisation* (Behr, Harel, Post & Lesh, 1994; Lamon, 1996) and *equivalence*, both of which are derived from the notion of base. It is binary in nature and therefore represents relational mappings (Halford, 1993). *Level 3 knowledge* is the structural knowledge that provides the superstructure for integrating all levels and is associated with *reunitisation*, *additive structure* and *multiplicative structure*. It incorporates ternary relations that are the basis of system mappings (Halford, 1993).

Within the model, *multiplicative structure* relates *position* and *base* into an exponential system (Behr, Harel, Post, & Lesh, 1994; Smith & Confrey, 1994) to give *value* and *order*. It is continuous and bi-directional and, for binary relationships, relates all adjacent places to the left through multiplication by 10 and to the right through division by 10. (For ternary relationships, it relates all adjacent-but-one places to the left through multiplication by 100 and to the right through division by 100.) It is the knowledge structure that underlies the concept of place value, the development of which is a major teaching focus in the primary school. Thus, an understanding of multiplicative structure is crucial and, as argued by Baturo (1997), if not explicated for whole numbers, denies students one of the major conceptual underpinnings of decimal numbers. It is also an excellent example of an abstract schema (Ohlsson, 1993) as shown in Figure 2.

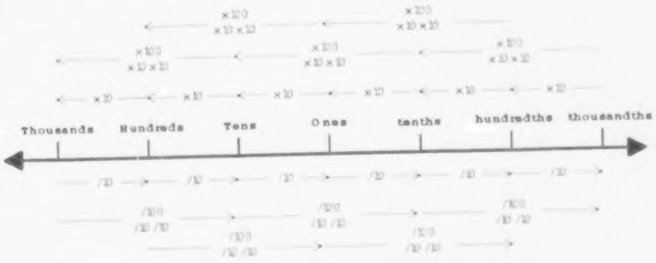

Figure 2. Place-value relationships embedded in the decimal number system.

The model was used by Baturo (1997) to develop interviews designed to probe stdents' understanding of Levels 1, 2 and 3 knowledge with respect to decimal numbers to hundredths. These interviews were administered to all students whose test performance was very high (≥90%), high (80-90%), and medium (60-80%). Thus, the interview selection comprised 16 very-higher performers (VHP), 16 high performers (HP) and 13 medium performers (MP). Responses to the interviews (and the tests) showed that a majority of the students did not have multiplicative structure to Level 3; in

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fact, a significant proportion of the medium students did not have multiplicative structure at Level 1 (knowledge of position and order). Therefore, intervention was undertaken, individually, with 17 of the 45 interview students to help them construct multiplicative abstract schema for decimal-number numeration.

The intervention study

Three types of intervention were given to the 17 students (1 VHP, 7 HP, 9 MP). Type 1 intervention was employed if the student had indicated evidence of procedural knowledge for interview tasks such as "0.3 × 10 = __". This intervention aimed to connect the student's procedural knowledge to the appropriate structural knowledge through focusing on reverse tasks such as "change 7 tenths to 7 ones using a calculator".

Students were given Type 2 intervention if Type 1 failed or if procedural knowledge was weak or unavailable. In Type 2 intervention used a large place value chart (PVC) and digit cards in conjunction with the calculator. The students were asked to model a binary relationship in one direction (×) by showing 7 tenths on the PVC, making a change to the 7 tenths to show 7 ones, and mirroring this process with the calculator. If students were successful on this task, they were then asked to model a ternary relationship (e.g., 8 ones to 8 hundredths) in the opposite direction (÷). The decimal point was represented with the students' choice of small adhesive stickers of hearts, geometric shapes, flowers or small animals. This was done to: (a) make the students aware that the decimal point, like all mathematical symbols, is a cultural artifact; (b) to add some excitement and motivation to an otherwise fairly dull task, and (c) to make the symbol more meaningful by allowing the students to choose their own representation. In this stage, the language used was vital in helping the students connect the concrete/iconic place value procedures to the symbolic calculator procedures.

Type 3 intervention was given to those students who, in Type 2 intervention, had shown an understanding of the bi-directional operations (×, ÷) that would effect the direction of the shift but who did not understand the role of the base in binary (adjacent places) relationships (and, therefore, ternary relationships). Students were shown the sets of whole-number statements below and asked which statement in each set was correct. The statements were chosen to be within the students' syntactic understanding for multiplying and dividing by 10 (i.e., to be solvable by invoking rules such as "add/take off a zero"). This task was used in conjunction with the PVC and digit cards to represent the multiplicative structure of whole numbers in order to transfer this knowledge to decimal numbers.

Set 1: 60 × 10 = 600, 60 ÷ 10 = 600, 60 + 10 = 600, 60 − 10 = 600;

Set 2: 800 × 10 = 80, 800 ÷ 10 = 80, 800 + 10 = 80, 800 − 10 = 80.

Results and discussion

Type 1 intervention. This intervention involved Claire (VHP) and Kylie (HP) who had exhibited robust procedural knowledge in the interview. They were encouraged to make the connection between place change and operation. Claire was asked to change 7 tenths to 7 ones using the calculator. She entered 7 tenths correctly, but her finger hovered over the + key and then over the 0 key. She finally shook her head and said: *I can't do it*. However, she very quickly made the connection when directed to examine her correct answers to the procedural tasks (e.g., 0.3 × 10 = 3), as the following protocol indicates. (Students' responses are in square brackets; I: = interviewer; S: = student.) *I: Here, we had 3 tenths multiplied by 10 equals 3 (pointing to each component of the procedural item). [S: Ohhh (immediately reaching for the calculator and entering × 10).]* On Kylie's first attempt, she entered "+ 7"; for her second attempt, she entered "7.00". She made one more attempt but then realised that that didn't work either. At this stage, she was given intervention similar to Claire with the same success.

Type 2 intervention. Each of the remaining 15 students were asked to show *7 tenths* on the place value chart and then to move the digit to show *7 ones*. They were then asked in which direction (right or left) they had moved the digit and whether the digit had become larger or smaller than it was before. Thus the students' kinaesthetic knowledge of position change was developed though moving the digit card whilst the associated language linked direction with size. The students were then asked to show this change on their calculator. This process was repeated for other adjacent places so that the relationship between the operation (× 10) and the leftwards direction was consolidated. Once the leftwards direction was associated with an increase in value, the students were asked to predict which way they would have to move the digit to effect a decrease in value and then asked to use their calculator to show the operation that would make the digit shift one place to the right. Again, the relationship between the operation (÷ 10) and the rightwards shift was consolidated with other adjacent places. The same process was repeated to establish the relationship between the operation, the direction of the shift and the number of places shifted with ternary relationships. However, although the continuous and bi-directional properties could be simulated and promoted through the place value chart (PVC) activity, the exponential property could not. So for those students who did not have an understanding of the role of the base, this activity was not effective. However, for those who did have the notion of the role of the base, this intervention seemed to have an immediate positive effect on making the appropriate connection between procedural and structural knowledge.

Once the students had moved the digits themselves (both directions) and then mirrored the processes required for both binary and ternary relationships, their new-found understanding was consolidated through activities where the interviewer moved the digit card (random direction and relationship but limited to ternary) whilst they mirrored the shifts on their calculator. Himansu's (MP) protocol exemplifies the language used throughout Type 2 intervention. *I: Show me 7 tenths on the place value chart. [He did so.] Now show me where you want to get it to show 7 ones. [He slid the digit card from the tenths place to the ones place.] Have you made the 7 larger or smaller in value? [S: Bigger] How many times bigger? [S: Ten times bigger.] Now enter 7 tenths on the calculator. [He did so.] What will you do to make the digit shift from the tenths place to the ones place? [He entered × 10 and was delighted to see that the operation produced the required shift.] Now, how do you think you could change the 7 ones back to 7 tenths? [He entered ÷ 10.] Well done.*

This stage of the intervention was repeated until he had shown a connection between ternary shifts to the left with multiplication (× 100) and to the right with division (÷ 100). The next stage of the intervention was then undertaken. *I: Now I'm going to move the digit (PVC) from there (7 tens) to there (7 hundreds). How can you do that on the calculator? [S: Multiply by 10.] So you make it one place bigger when you multiply by 10. How do you think we could get the 7 hundreds back to 7 ones (showing on the PVC)? Is it getting larger or smaller in value? [S: Smaller. (He entered – 100 and had 600.) No, that's wrong.] What undoes multiplication? [S: Divide.] Well, leave your 6 hundreds and make it into 6 ones (showing on the PVC). [He divided by 100.] Now, watch carefully because I'm going to try to catch you (shifting the PVC digit from 6 ones to 6 hundredths). [He divided by 100.] Excellent. How did you know to divide by 100? [S: Because 10 times 10 is a hundred.] Well done! And did you get larger or smaller when you went from there to there (indicating ones to hundredths on PVC)? [S: Smaller.] One more go but I'm not going to say anything so you have to watch what I do (placing the digit, 3, in the tenths place and moving it to show 3 tens). [He entered × 100, looking very pleased with himself.] What a champ!* The success experienced by Himansu was particularly gratifying as he had been totally unsuccessful on the interview tasks related to position and order . His body language during the intervention changed from what apparent nervousness to confidence whilst his smiles indicated that this intervention had boosted his self-esteem.

This stage of intervention was very successful for 8 of the 15 students. Of the remaining 7 students, 5 eventually associated the leftwards shift with multiplication for both binary and ternary relationships but continued to associate the rightwards shift with subtraction. These students knew the equivalence relationship of 10 between adjacent places and the relationship of 100 between adjacent-but-one places but were unable to connect the relationship to multiplicative operations. Kirsty's protocol exemplifies the difficulties in eliciting the connection between equivalence and the required operation. *I: Show me 7 tenths on the place value chart. [She did so.] Now show me where you want to get it to show 7 ones. [She slid the digit from the tenths place to the ones place.]*

Have you made the 7 larger or smaller in value? [S: Larger] How many times larger? [S: 10] Show me on your calculator how to change 7 tenths to 7 ones. [She entered 0.7 and then entered 10; she was bewildered when she saw the result, 0.71.] How many times larger than 7 tenths is 7 ones? [S: Ten times larger. (Her finger hovered over the + key but she didn't press it.] What can you do to tenths to get ones? [She entered + 10 and again was bewildered by the result, 10.7.] What else could you do? [No response] You made the 7 tenths 10 times bigger here [PVC], didn't you? [S: Yes] So what else could you do apart from adding 10 to shift 7 tenths to 7 ones? [S: Times by 10?] Try it. [She entered × 10 and looked very pleased with herself when she saw the result.] Now, I'm going to shift the 7 ones back to 7 tenths (showing on PVC). How can you make the calculator do that? [She entered − 10!]

For all students who could not connect equivalence with the multiplicative operations, the following questions usually elicited the given responses. *How many tens equal a hundred? [10] How many times larger than tens are hundreds? [10 or 10 times larger] What can you do to tens to get hundreds? [Add 10; add 90; × 10 (not often)] What can you do to hundreds to get tens? [Subtract 10; subtract 90; ÷ 10 (not often)]* Thus, the first two questions elicited the base (10) but not the operation, whilst the last two questions elicited an operation which, for most lower-performing students, will be addition and subtraction (additive structure) or multiplication and subtraction (conflict between multiplicative and additive structure). The latter response, giving the multiplication operation but not the division operation, may have been the result of the word "times" in the previous question. The students with this type of problem were not provided with the third type of intervention because they already had an awareness of the base. However, although the consolidation activities helped these students, it was thought that they would require other, more intensive, remediation to establish the connection between equivalence and multiplicative operations and to develop the notion of division as the inverse of multiplication.

The remaining 2 students, Dean and Sarah (both HP) revealed that they had associated the appropriate operations with the bi-directional shifts but they were not aware of the role of the base in binary and ternary relationships. These two students were given Type 3 intervention.

Type 3 intervention. This intervention initially focused on the binary patterns in Set 1 of the mathematical statement (i.e., 60 × 10 = 600, 60 ÷ 10 = 600, 60 + 10 = 600, 60 − 10 = 600). Instruction followed this sequence of steps: (a) the students' attention was drawn to the similarities between the starting and finishing numbers (i.e., 60 and 600); (b) they were asked to show 6 tens on the PVC and then shift the 6 to its finishing position; (c) they were asked to select the operation from the list of statements that would make that shift; (d) they were asked to show the shift from 6 tens to 6 hundreds on the calculator; (e) they were asked to show similar binary multiplication shifts for other adjacent places (e.g., 7 tenths to 7 ones; 5 hundredths to 5 tenths); and (f) they were asked to use the calculator to show ternary multiplication shifts that were shown on the PVC (e.g., 8 tens to 8 thousands). These steps were followed for the second set of statements to extend the role of the base in binary and ternary relationships to division. This intervention, in combination with Type 2 intervention, was successful for both Dean and Sarah.

Implications

This study indicated that students need the three levels of knowledge shown in the numeration model (see Figure 1) to understand and access multiplicative structure. The students within the study exhibited particular weaknesses with regard to the bi-directional nature of multiplicative structure and, whilst their responses indicated a knowledge of base and equivalence, they were unaware of the role of base and equivalence in linking decimal number places. Moreover, the students' knowledge appeared to be available in static conditions only (connecting two given places) and was generally not translated to dynamic conditions in which "10 times larger" needed to be associated with a shift one place to the left and "10 times smaller" needed to be associated with a shift one place to the right. Students were able to apply the exponential relationship more successfully within the domains of whole numbers or decimal numbers (e.g., tens and hundreds, ones and tenths) than across these domains (to nonprototypic examples such as tens and tenths), and between adjacent positions (e.g., tens and hundreds) than between non-adjacent positions (e.g. tens and thousands). There was also evidence that whole numbers had been introduced with a focus only on grouping (multiplication) by ten and the new decimal positions with a focus only on partitioning (division) by ten. Thus, teaching should: (a) facilitate knowledge and integration of the three levels of knowledge; (b) give priority to the bi-directional relationship, starting with whole numbers and extending to decimal numbers; (c) use a dynamic approach to change; (d) use examples across domains and between non-adjacent positions; and (e) include *reverse* activities, that is, partitioning for whole numbers and grouping for decimal numbers.

The success of the interventions indicated that students need to experience material usage that reinforces size and bi-directional relationships. Therefore, grouping material such as MAB should be used to show the size of a ten in comparison to a hundred and to show the bi-directional relationship (i.e., a ten is 1 tenth of a hundred; a hundred is equivalent to 10 tens); and partitioning material such as 10 × 10 grids (a square divided into 100 smaller squares in 10 rows of 10) should be used to show the size of 1 tenth compared to 1 hundredth and the bi-directional relationship (1 tenth is 10 hundredths; 1 hundredth is a tenth of a tenth).

Place value charts show position and order effectively and efficiently but do not show size in a concrete way and, by themselves, do not show the exponential relationships nor the effect of applying such relationships. Calculators, on the other hand, do show the effect of applying exponential relationships. The actions of the students revealed that modelling with both place value charts and calculators simultaneously had internalised the place value chart as an exponential model rather than as a simple positional model. For example, some students nodded their heads twice as they mentally moved from tenths to tens (for example) while others indicated with their fingers that they were moving across two places. Therefore, activities which require the students to physically move digits from one place to another on the place value chart appear to develop the kinaesthetic aspect of the exponential relationship whilst the calculator verifies the operation that effects the shift in position and together, they provide a connection from external representations to internal representations. Thus, the interventions indicated that place value charts, in tandem with calculators, are invaluable aids in showing position, order and the bi-directional nature of exponential relationships.

Although successful at the time, it is doubtful whether the interventions would have long-term effects for those students who could not connect the notion of equivalence with the multiplicative operation or who were unaware of the role of the base in binary and ternary relations but connected the direction with the multiplicative operations. Students such as these would need to have further intervention to establish the appropriate notions and to connect these notions to the exponential model.

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CLASSROOM-BASED RESEARCH TO EVALUATE A MODEL STAFF DEVELOPMENT PROJECT IN MATHEMATICS[*]

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Abstract

This paper reports the results of an observational study of fourteen high school mathematics teachers who had been involved in varied numbers of years of professional development focusing on content, pedagogy, equity issues, and use of technology. Data from the study include classroom observations, informal and formal interviews with teachers, and artifacts such as assessments, student projects, portfolio assignments, computer/graphing calculator lab activities, and student journal assignments. These data were analyzed using qualitative methods, with categories formed by patterns discernible across the types of data collected. We concentrate specifically here on issues of teachers' subject matter knowledge and pedagogy.

Focus of the Paper

The main purpose of this project was to investigate the effect of two professional development projects for high school teachers. One of these projects (funded by the Eisenhower Mathematics and Science Education State Grant Program) involved teachers in a single year of staff development, while the subsequent project (funded by the National Science Foundation) involved teachers in two to three years of intensive professional development. Both projects focused on content, pedagogy, attitudes, equity and leadership. We were interested in two major questions:

What is the relationship between the professional development and changes in classroom practice?

What differing effects did the various levels of staff development have on teachers' classroom practices?

Although both investigators observed all teachers in the study, they each had different major foci for observations and interviews. One observer focused on issues of gender/ethnic equity and how these were manifested in teacher-student interactions, student-student interactions, and curriculum choice. This was of particular interest because both projects worked with schools which were eliminating tracking as a means of increasing the number of underrepresented students in academic courses. The second observer focused on how expanding views of algebra and geometry were integrated into the content and pedagogy. In particular, she was interested in how technology and modeling had become an integral part of the classroom instruction.

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Theoretical Framework

The professional development projects upon which this research paper is based were founded on the assumption that what a teacher believes and what a teacher knows both influence the teaching of mathematics (Fennema & Franke, 1992; Thompson, 1992). Here what a teacher knows is understood to include both content knowledge and pedagogical content knowledge (Cooney, 1994). Cooney (1994) has interpreted Shulman's (1986) original notion of pedagogical content knowledge in the discipline of mathematics. For Cooney, pedagogical content knowledge in mathematics involves integrating content and pedagogy, borrowing ideas both from mathematics itself and from our knowledge about teaching and learning mathematics. He presents the example of the rational numbers, for which we have various interpretations and a deep knowledge base about how children construct their understanding of the rational numbers through these different interpretations. This integration of the mathematical and psychological domains defines pedagogical content knowledge in this domain.

As we structured the inservice education to enhance both content and pedagogical content knowledge, we were mindful that teachers themselves are constantly constructing knowledge, albeit knowlege about students' learning of mathematics, effective teaching of mathematics, as well as mathematical content. Therefore activities were structured to ensure that knowledge was actively developed by participant teachers, not passively received. Participants were frequently involved in presentations, facilitation of small group activities, and even development of workshop foci. The professional development became a collaboration among university faculty, district curriculum coordinators, and participant teachers.

Related Literature

In the last edition of the *Handbook of Reseach on Teaching*, the chapter on mathematics education (Romberg & Carpenter, 1986) hardly mentions research on inservice teacher education. As Grouws pointed out (1988), and as is still the case, there is little information available about the overall design features of inservice education programs which maximize changes in teacher beliefs and ultimately classroom practices. Grouws called for studies which focus on the impact of various features of inservice education on classroom practice.

Cooney (1994), in a review of research and teacher education, notes that, while we are collecting in the literature many insightful stories about the lives and work of teachers, we have yet to move beyond to develop theories which can help explain what we see and predict what effects teacher education will have. Cooney offers one such theoretical perspective, what he calls authority, derived from the work of Perry (1970) and the feminist conceptualization of ways of knowing developed by Belenky, Clinchy, Goldberger and Tarule (Belenky, Clinchy, Goldberger & Tarule,1986, 1997; Goldberger, Tarule, Clinchy & Belenky, 1996). In *Women's Ways of Knowing* (Belenky, Clinchy, Goldberger & Tarule, 1986, 1997) for example, there are ways of knowing which are bound to external authorities as the source of all knowledge and correct answers. Until a teacher begins to see authority as an internal agent, Cooney points out, that teacher cannot accept the relativism and sense of context necessary to exert control of curriculum and even pedagogy. Current reform movements in mathematics in the USA call for teachers to be reflective, adaptive and have a constructivist orientation. Cooney claims that such an orientation cannot be achieved if one views the world in general, and the teaching of mathematics more specifically, in absolute terms.

Methods

In previous small scale research on these projects, we had to rely primarily on self-reported data, both quantitative and qualitative, from participant teachers. We felt it essential to complement these studies with extensive observations of classes, perusal of auxiliary materials such as assessments, and pre- and post-observation interviews with participant teachers.

Thus we designed a qualitative study using participant observation techniques (McCall & Simmons, 1969). Our mode of working in the classroom included a pre-interview (often informal) about the goals of the lesson. During lessons we often interacted with students as they worked collaboratively in groups or on individual tasks, asking and sometimes answering questions. After observed lessons, a debriefing interview was held in which we discussed with the teacher: decisions s/he made concerning ways to teach the specific content; how s/he perceived the lesson succeeded in meeting goals; and, how this lesson fit into the sequence in the unit. Ancillary materials were collected for later examination, including worksheets, tests and quizzes, student projects, portfolio instructions, computer and calculator labs, and journals.

Each teacher was observed weekly over a semester, with each investigator alternating visits. This enabled the two to highlight questions of interest for the other to pursue in her next visit.

Unstructured observational field notes were collected during observations. Interviews were mostly informal, with notes taken during the interview and expanded immediately afterward. All notes were transcribed and expanded, with patterns and questions to investigate further identified as work progressed (Glaser & Strauss, 1967). The aim is to provide a rich description of the classroom practices and how they have been affected by the inservice education in which teachers participated, leading to grounded theory.

Fourteen high school mathematics teachers were observed during the 1996-97 academic year. These teachers were chosen to represent different levels of involvement in inservice: one, two, three or four years. Although 24 were asked to participate in the study, only 14 were able to do so. The participants represented one with one year of inservice, two with two years, seven with three years, and four with four years. The sample included five male and nine female teachers from nine schools and four different school districts. Three were Asian and the rest European-American.

A total of over 200 classes were observed. Courses observed varied from an algebra restart [for students who were unable to succeed in algebra the first semester] to algebra 2/integrated course 3. A total of 17 different classes of the 14 teachers were visited, including one algebra restart, six algebra 1 classes, 5 geometry classes, one algebra 2, one integrated course 1, two integrated course 2, and one integrated course 3. The texts used varied from the traditional to "transitional," to integrated ones, as shown in table 1 below.

	Traditional	**Transitional**	**Integrated**
Course			
Algebra 1/Course 1	2	5	1
Geometry/Course 2		5	2
Algebra 2/Course 3	1		1

Table 1
Types of Textbooks Used for Each Course

This range indicates the curricular differences across the districts involved in the study. However, one commonality across districts is the elimination of tracking and placement of all ninth grade students in algebra 1/course 1 or a higher course.

Thus data from the study included observational field notes, interview transcripts and classroom artifacts.

Results

We have two categories of results to discuss in this paper: patterns in teachers' subject matter knowledge, and teachers' use of pedagogy. The specific topics within these categories are especially relevant as they were focal points of our workshops.

Content. In the category of content, we focused both on teachers' professed or inferred understanding of the content they had to teach and on how they chose to present topics difficult for students to understand.

- Some teachers are completely unfamiliar with content that they are being asked to teach in the new books.

One example is fractal geometry, included in the third year of *College Preparatory Mathematics: Major Change from Within* (Sallee & Kysh, 1990). One of the teachers in the study, who had been in four years of inservice, was faced with teaching a unit including fractal geometry. Although we had had several sessions on chaos and fractals, including Robert DeVaney as a speaker, and had bought fractal software for each school, Belinda expressed concern about her lack of knowledge in this area. However, later in the academic year Belinda volunteered to present a workshop on fractals using what she had learned from the internet and her experiences in teaching the unit.

Not all teachers, however, took this initiative to fill in gaps in their mathematical understanding. Charles, in an interview with the two researchers, expressed concern about his teaching of a new Advanced Placement statistics course; he felt unprepared mathematically for teaching the content. When we asked what he was doing to "get up to speed," he indicated he really did not have time to pursue this so was just keeping a day ahead of the students by reading the textbook.

- While we were able to purchase class sets of graphing calculators for each school, and spent considerable time addressing uses of technology in workshops, the potential integration of the power of technology in the teaching of mathematics has not been tapped in most classrooms.

Of the fourteen teachers, only one (Sharon) was observed to make extensive use of a computer lab in geometry. Four others used graphing calculators a number of times. Charles, who was teaching algebra 2, only used calculators once although there were a number of lessons in which they would have been appropriate. Christa began using calculators but stopped when one was stolen. The classes in which the calculators were used most extensively were taught by teachers who had curricular materials conducive to their use and who had had extensive inservice on technology. However, if teachers had to devise lessons and activities themselves to integrate the use of calculators or comptuers, this was much less likely to happen.

- Topics in probability continue to cause conceptual difficulty for some teachers, so that instruction in this important strand is diminished.

- Statistical concepts are also very difficult for teachers to teach for understanding. Reliance on algorithmic approaches to concepts such as variance and standard deviation does not aid students in understanding what these mean.

One example was in Polly's integrated course 1, in which she was introducing the concept of standard deviation. This was done initially in a rather traditional way, putting an extensive table on the board including these columns:

x values	$x-\mu$	$(x-\mu)^2$

then proceeding with the calculation. Polly made no attempt to provide understanding of what the standard deviation was measuring. In fact, after the class she volunteered to the observer that students would not know what the standard deviation means from her instruction, and she needed to think about how to do that. She did follow this exercise with use of the statistical features on the calculator so that students would not have to do the calculation by hand. In a later visit, discussion with students in groups confirmed that they could not describe in their own words what the standard deviation is.

However, in a later lesson, Polly did a nice activity, collecting data on the heights of students' navels from the floor and while standing on a chair, comparing various measures of central tendency. Students were not surprised that the mean changed but very surprised that the standard deviation did not. This activity seemed designed to help students begin to develop some intuition about these statistical concepts.

- Topics in discrete mathematics, such as applications of graph theory, are still relatively unfamiliar to many teachers.

Pedagogy. The inclusion of many more students in college preparatory mathematics has resulted in more heterogeneity in teachers' classes. Cultural and language diversity contribute to differences in learning styles which must be accommodated. While many of the teachers used cooperative learning, few of them made extensive use of: teaching and assessing problem solving; protracted projects; a balance of conceptual understanding, problem solving and skills; integrating mathematics with other disciplines; and techniques for working with English language learners.

Teachers in the study were very aware of issues of equity with respect to the "algebra for all" requirement. Support of this policy was strong among almost all of this sample. However, classroom inequities were still apparent in many classes. For example, teachers seemed unaware of bias which may be inherent in "traditional" forms of assessment. Even more striking were typical gender interaction patterns, with males dominating the classroom conversation, in many lessons observed. In addition, a small number of males dominated the conversation in these lessons. Ethnic differences were not so striking, perhaps because these classes had a large percentage of students of color.

An additional finding relevant to pedagogy is that teachers are struggling to use technology effectively throughout the curriculum. Questions teachers have include: sequencing, e.g., when does one introduce the technology in relationship to a certain concept; access issues, e.g., when should students have access to technology; the role of technology in mathematical modelling; and the changing view of algebra and geometry in a dynamic environment.

Summary

While the professional development projects had some substantial impact on the teachers, including movement away from traditional textbooks toward use of more "reform" texts, many of them are still struggling with the heterogeneity in classes due to detracking, with the full integration of technology, and with teaching new content in new ways. Support for teachers back in their classrooms as they attempt to change is critical. This research itself played an important professional development role, teachers reported, in that it enabled them to reflect on what they were teaching, how, and why with a non-evaluative observer. Perhaps in retrospect that has been the most important aspect of the staff development.

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SOME MISCONCEPTIONS UNDERLYING FIRST-YEAR STUDENTS' UNDERSTANDING OF '*AVERAGE RATE*' AND OF '*AVERAGE VALUE*'

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A research method consisting of written tests and individual interviews was introduced to explore first-year university students' understanding of fundamental calculus concepts, after the concepts had been dealt with in their first-year calculus course. A total of 630 students from three South African universities were subjected to the tests pertaining to this study. The analysis of written and verbal responses to diagnostic test items revealed significant information regarding the nature and characteristics of students' concept images for key calculus concepts. Several erroneous conceptions underlying students' mathematical activity, and some errors that originated from it, were identified. This paper deals with students' understanding of average rate, average value of a continuous function and average velocity.

1. Introduction

Many mathematics educators realise that a large proportion of students have great difficulty in grasping key calculus concepts such as limits, continuity, derivative and integral [Ferrini-Mundy and Graham, 1991]. It seems that the origin of students' difficulties with these concepts is more profound than is often anticipated. Remarks like the following indicate that the learning and teaching of the calculus need appropriate attention: '*... the state of most students' conceptual knowledge of mathematics after they have taken calculus is abysmal*' [Epp, 1986 : 48]. '*Students demonstrate virtually no intuition about the concepts and processes of calculus*' [Ferrini-Mundy and Graham, 1991 : 631].

The main purpose of the research project, on which this paper is based, was to gain more information on students' understanding of basic calculus concepts, after the concepts concerned had been dealt with in the first-year calculus course. This paper, however, is concerned only with students' understanding of '*average rate of change*' and '*average value of a continuous function*'.

2. Theoretical Background

The theoretical framework for the research under discussion was developed according to the principles of the theory of constructivism. A central idea of the constructivist theory is '*that understandings are constructed by learners as they attempt to make sense of their experiences; each learner bringing to bear a web of prior understandings, unique with respect to content and organization*'. [Simon and Schifter, 1993 : 331]. Within this theoretical perspective the idea of a concept image (a term adopted from Tall and Vinner [1981 : 151]) was used. The term concept image refers to the total cognitive structure that exists in an individual's mind regarding all aspects of a specific mathematical concept. This is a structure that an individual creates and develops as a result of personal experiences with a concept.

3. Method

The method used to explore students' concept images consisted of three phases. In the first phase 107 engineering students wrote the preliminary tests. These tests were conducted during a regularly scheduled tutorial class at the end of the second semester of 1994.

The final diagnostic tests, consisting of Diagnostic Test A and Diagnostic Test B, were compiled after analysis of the results obtained from the preliminary tests. A total number of 523 first-year university students from three South African universities participated in the final testing that was conducted near the end of the second semester of 1995. This group included students in engineering, the physical sciences and students enrolled in service calculus courses. For the analysis of test results, a random sample of 100 answer-books for each of the two tests, was taken from the three different groups that participated.

The third phase involved task-based interviews with 15 students who had written both final tests. For administration purposes the students were numbered S_1 to S_{15}. Each of these students participated in two one-hour interview sessions. The interviews were structured around specific test items selected from the final tests. All interviews were audiotaped.

4. Test items concerning average rate and average value

A selection of some of the test items concerned with *'average rate'* and *'average value of a continuous function'*, or aspects thereof, appears in the appendix to this paper. This article deals mainly with students' responses to the following test items: test items 6.1 and 6.3 (Diagnostic Test A); question 2 (Diagnostic Test B). We would like to make some remarks concerning these three items.

4.1 Test items 6.1 and 6.3: Together the two items bring out two natural interpretations regarding the average rate of change. Considering that $S(x) = g'(x)$ the symbolic equation $\dfrac{\int_0^3 S(x)\,dx}{3} = \dfrac{g(3) - g(0)}{3}$, which indicates the equality of the two averages, therefore holds the idea that the average value of $g'(x)$ on $[0, 3]$ is equal to the average rate of change of $g(x)$ with respect to x on $[0, 3]$.

4.1 Question 2: A variety of function representations were used in the two diagnostic tests. While the two functions in question 6 have a tabular representation, the function v in this question is defined in algebraic symbols. In question 1 of

Diagnostic Test B the function representation is graphic (see appendix). The main reason for the utilization of different representations of functions was to explore students' understanding of concepts within different modes of representation.

This test item contains mathematical content that is closely related to that of test items 6.1 and 6.3. If, in this question, the distance travelled in t seconds is denoted by $s(t)$, then the symbolic equation $\frac{s(4)-s(0)}{4} = \frac{\int_0^4 v(t)\,dt}{4}$ indicates that the average rate of change of distance with respect to time over the time interval [0, 4] is the same as the average value of the velocity function over the time interval [0, 4]. The principal aim of this test item was to gain more insight into students' concept images for average velocity.

5. Students' procedures and conceptions

The analysis of students' written and verbal responses revealed significant information regarding the nature and characteristics of students' understanding of fundamental calculus concepts. Several misconceptions underlying students' activity in the calculus, as well as errors originating from the application of such misconceptions, were identified. In this section the emphasis is on some of the erroneous procedures and misconceptions pertaining to students' thinking of average rate and of average value of a function.

We will first consider some responses of students to test items 6.1 and 6.3 of Diagnostic Test A and then responses to question 2 of Diagnostic Test B (see the appendix for the test items). The following table shows the distributions of students' answers to question 6.

TABLE 1

Distribution of answers to test items 6.1 – 6.3

(Number of students = 100)

Test item	6.1	6.2	6.3
Correct	23	48	3
Incorrect	63	41	77
No Answer	14	11	20

(Only one student answered both 6.1 and 6.3 correctly).

Students' attempts to find the average rate of change in test item 6.1 and the average value in 6.3 resulted in a variety of procedures being used. Table 2 below contains a summary of some of the procedures for the two test items. The number of students who applied a specific procedure is also indicated. For the purpose of discussion, the procedures in the table are numbered. Since it was given that the two functions in this question is such that $g'(x) = S(x)$, we assumed that students that drew upon function values of S in their answers to test item 6.1, had in mind function values of g'. (Many of these students, but not all of them, did indeed refer to function values of g' in their solutions).

TABLE 2
Some procedures concerning test items 6.1 and 6.3

	Procedures for 6.1	Procedures for 6.3
1	$\frac{g'(0)+g'(0,5)+g'(1)+g'(1,5)+g'(2)+g'(3)}{6}$ (10 students)	$\frac{S(0)+S(0,5)+S(1)+S(1,5)+S(2)+S(3)}{6}$ (25 students)
2	$\frac{g'(3)+g'(0)}{2}$ (2 students)	$\frac{S(3)+S(0)}{2}$ (5 Students)
3	$\frac{g'(3)+g'(0)}{3}$ (2 students)	$\frac{S(3)+S(0)}{3}$ (3 students)
4	$\frac{g'(3)-g'(0)}{3}$ (3 students)	$\frac{S(3)-S(0)}{3}$ (10 Students)
5	$\frac{g'(3)-g'(0)}{2}$ (1 student)	$\frac{S(3)-S(0)}{2}$ (1 student)
6	$g'(3)-g'(0)$ (6 students)	
7	$\frac{g(0)+g(0,5)+g(1)+g(1,5)+g(2)+g(3)}{6}$ (3 students)	
8		$\frac{S(0)+S(0,5)+S(1)+S(1,5)+S(2)+S(3)}{5}$ (3 students)

The interviews on test items 6.1 and 6.3 led to the disclosure of various aspects of students' conceptions regarding average rate and average value. During the interview sessions students were encouraged to give reasons for procedures they had applied. Student S_4 made the following remark with respect to his application of procedure 2 for test item 6.3 (see table 2): '*When I see average value I immediately want to add the two together and divide them by* 2'. It is evident that this student's concept image for arithmetic mean had come into play here.

During interview sessions students' attention was often drawn to the two averages in 6.1 and 6.3. After student S_4 had compared test item 6.1 and 6.3 with each other, he expressed his opinion: '... *So, in words, those two questions are actually the same thing. Although I can understand why they are not the same. Okay, I can't really. I know they shouldn't be the same, but it makes sense to me if I say they are the same.*' At a later stage during the interview session the student mentioned that '... *they are* *different answers, but they ask, I am sure they ask the same thing'.* It seems that this is an instance where the erroneous procedure dominates the student's conceptual understanding of a mathematical situation.

Student S_1 included the following in his explanation for applying procedure 5 (test item 6.3): *'If you want the average, in other words you then want to find the average rate of change. You then take the function value at the endpoint of the interval minus the function value at the initial point of the interval and divide it by 2. Then you get the average value at which it changes'**. This student's reasoning points to a conception of average value that emanated from a distorted concept image for average rate of change. Student S_{12} explained that his procedure $\frac{S(3)-S(0)}{3}$ (procedure 4) for test item 6.3 is based on the *'idea'* that the average value of $S(x)$ is equal to *'the change in $S(x)$ over the change in x'*. Information that was gathered from students' explanations suggests that some procedures may be the result of misconceptions that originated from an interspersion of knowledge of different concepts, including *'arithmetic mean'* and *'average rate of change'*.

As was the case with the averages in test items 6.1 and 6.3, many students did not deal successfully with the average velocity in question 2 of Diagnostic Test B. Only 10 of the 100 students answered it correctly, while 88 of the students gave incorrect answers. Two of the students did not answer this item. Students' erroneous procedures include the following:

- $\frac{v(4)-v(0)}{4}=3$ (12 students)
- $v'(t)=\frac{3}{2}t$
  $v'(4)=6$
  Average velocity = 6 (9 students)
- $\frac{v(4)}{4}=3$ (6 students)
- $\frac{v(4)-v(0)}{2}=6$ (7 students)

The procedures above do not reflect average velocity as the ratio $\frac{\text{distance travelled}}{\text{time elapsed}}$. There were some other procedures with the same deficiency. During interviews it was found that such procedures were often the result of a more mechanical approach and that the application of relevant conceptual aspects was lacking. Student S_9 for example explained that his procedure, $\frac{v(4)-v(0)}{4}$, shows that the average velocity is '*the total change in y over* 4 *seconds divided by the total change in x for the* 4 *seconds*". He added that the y denotes the function values of v and that x corresponds with t. Unlike student S_9, the student with a mature concept image for average velocity will take conceptual aspects, like the above-mentioned ratio, into account in such a situation.

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* An asterisk appearing next to a quotation, indicates that the quotation has been directly translated from Afrikaans to English.

5. Conclusion

The research described in this paper has proved fruitful in revealing some of the ways that calculus students think about average rate and average value. A conceptual deficiency demonstrated by students participating in this study, was inadequate intuition about the two concepts. If students can enrich their concept images for these concepts it may enhance their understanding of the Mean Value Theorem and the Fundamental Theorem of the calculus – the concepts mentioned are closely related to these theorems.

For the average calculus student it may take an extended period of time to develop a satisfactory understanding of the rate of change concept. Meaningful experiences with the average rate of change concept at an early stage at the secondary school level (before students' introduction to a formal calculus course) may help in this regard. Moreover, the rate concept needs to be revisited at various times during the calculus course. If elements of a concept image are not constantly reinforced, it has a good chance to deteriorate and thus become distorted [Vinner, 1983 : 305].

Findings of this study suggest that students should get adequate opportunities to deal with calculus concepts, in various representations and in conceptually based situations. Meaningful experiences of calculus concepts in graphical and numerical representations can make an important contribution to the student's understanding of the conceptual underpinnings of the calculus.

APPENDIX

DIAGNOSTIC CALCULUS TEST

TEST A

QUESTION 6

The following table shows certain x-values and the corresponding function values of two continuous functions S and g which are such that $g'(x) = S(x)$:

x	-1	-0,5	0	0,5	1	1,5	2	3
$S(x)$	4	0,75	-1	-1,25	0	2,75	7	20
$g(x)$	0	1,125	1	0,375	0	0,625	3	16

Find (show all calculations):

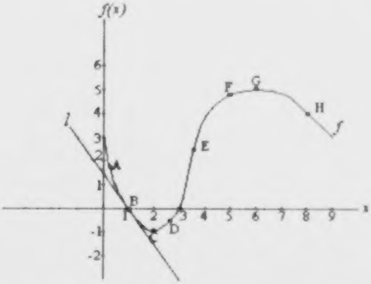
6.1. the average rate of change of $g(x)$ with respect to x over the interval $[0, 3]$;

6.2. $\int_0^3 S(x)\,dx$;

6.3. the average value of $S(x)$ for $0 \leq x \leq 3$.

TEST B

QUESTION 1



The figure shows the graph of a function f for $0 \leq x \leq 9$ with l a tangent line to the graph of f at the point (1; 0). Use the graphic representation of the function f to answer the following.

1.2. For which one of the points A, B, C, D, E, F, G or H is $\dfrac{f(x+0{,}002)-f(x)}{0{,}002}$ closest to 1.

1.4. Between which pair of consecutive points on the graph is the average rate of change of $f(x)$ with respect to x the greatest? Choose, therefore, only one pair from the following seven pairs of consecutive points:
A and B; B and C; C and D; D and E; E and F; F and G; G and H.

1.6. Determine the value or an approximation thereof (as accurately as possible) for each of the following:

1.6.4. $\int_{5}^{9} f(x)\,dx$.

1.6.5. The average value of $f(x)$ for $5 \leq x \leq 9$. Also show your calculation of this average value.

QUESTION 2

A vehicle, initially at rest, accelerates so that its velocity v after t seconds is given by $v(t) = \frac{3}{4}t^2$ metres per second for the first 4 seconds.

Calculate the vehicle's average velocity for the first 4 seconds. (Show your calculations).

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Operable Definitions in Advanced Mathematics: The Case of the Least Upper Bound

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This paper studies the cognitive demands made on students encountering the systematic development of a formal theory for the first time. We focus on the meaning and usage of definitions and whether they are "operable" for the individual in the sense that the student can focus on the properties required to make appropriate logical deductions in proofs. By interviewing students at intervals as they attend a 20 week university lecture course in Analysis, we build a picture of the development of the notion of least upper bound in different individuals, from its first introduction to its use in more sophisticated notions such as the existence of the Riemann integral of a continuous function. We find that the struggle to make definitions operable can mean that some students meet concepts at a stage when the cognitive demands are too great for them to succeed, others never have operable definitions, relying only on earlier experiences and inoperable concept images, whilst occasionally a concept without an operable definition can be applied in a proof by using imagery that happens to give the necessary information required in the proof.

Mathematicians have long "known" that students "need time" to come to terms with subtle defined concepts such as limit, completeness, and the role of proof. Many studies have highlighted cognitive difficulties in these areas (e.g. Tall & Vinner, 1981, Davis & Vinner, 1986, Williams, 1991, Tall, 1992). Some authors (e.g. Dubinsky et al, 1988) have focused on the role of quantifiers. Barnard (1995) revealed the subtle variety in students' interpretations of statements involving quantifiers and negation. Nardi (1996) followed the development of university students' mathematical thinking by observing and audio-taping their first year tutorials. This highlighted the tension between verbal/explanatory expression and formal proof, and tensions caused in proofs by quoting results of other theorems without proof. Vinner (1991) drew attention to two modes of use of definition – the everyday, and the technical mode required in formal reasoning. In this paper we report a longitudinal study of the individual developments of students in their first encounter with a formal mathematical theory, to see the growth of their use of definitions in building concepts and proving theorems.

We initially formulated the following working definition:

> A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument.

The intention of the study is to track the construction of the concept of least upper bound, and related concepts such as continuity, to see how (or if) operability develops over time and how this relates to the use of the concepts in later theorems such as the existence of the Riemann integral.

Methodology

Five students were selected to be interviewed (and video-taped) on five occasions during a twenty week Analysis course consisting of sixty one-hour lectures. This followed a long-established syllabus in which the axioms for a complete ordered field were given in the third lecture (using the least upper bound form of the completeness axiom) followed by the convergence of sequences and series, the continuity and differentiability of real functions, the Riemann integral and the Fundamental Theorem of Calculus.

The interviews were semi-structured, in that the same lists of questions were used with each student but were then followed up in response to the students' answers. Students were invited to speak about their experiences of the course and beliefs and attitudes, as well as answering more directly content-related questions. Each interview tried to capture the state of development of the student at that time, focusing initially on recently covered work, then checking on the longer-term development of selected fundamental conceptions including the notion of least upper bound.

Of the five students, Lucy, Matthew and Martin were mathematics majors whilst Alex and Sean were physics majors taking several mathematics courses, including Analysis. Lucy proved to be the most successful of the five; she was invariably able to talk coherently about concepts and theorems but did not memorise definitions, preferring to draft what she knew on paper and then refine her ideas. Matthew worked very hard, attempting to commit material to memory by repeated readings; when explaining things he would sometimes break down and then need to "refresh his memory" by looking at his notes. Alex missed more lectures than the others and was not always conscientious in copying up the notes; he later changed courses without taking the end of year exam, nevertheless he had certain ways of operating which will be central to the discussion which follows. Both Martin and Sean found the requirement for formality bewildering and were unsuccessful in their examinations. We therefore choose to focus on the work of Alex and contrast this with the more successful Lucy and the less successful Sean.

First encounters with the definition of a least upper bound

In the third lecture, the following definitions were given:

> An *upper bound* for a subset $A \subset \mathbb{R}$ is a number $K \in \mathbb{R}$ such that $a \leq K \ \forall a \in A$.
>
> A number $L \in \mathbb{R}$ is a *least upper bound* if L is an upper bound and each upper bound K satisfies $L \leq K$.

In the early interviews, all the students showed that they could give the concept some kind of meaning, varying in the relationship to the formal definition. Lucy and Alex focused on the second part of the definition which does not explicitly define the notion of upper bound whilst Sean used his own imagery:

> *Lucy*: Well, say k is an upper bound for the set, then we'll say that m [the least upper bound] is less than or equal to k.
>
> *Alex*: A least upper bound is the lowest number … that is an upper bound. Any number greater than it, no matter how little amount by, it's not going to be, you know it's not going to be, in the set.
>
> *Sean*: The supremum of a set is the highest number in the set.

Lucy and her struggle giving meaning to the least upper bound

By the second interview Lucy was able to verbalise the definition of least upper bound in a manner close to the symbolic form:

> *Interviewer*: If I asked you what was a least upper bound what would you say now? What properties would you say that that's got?
>
> *Lucy*: Well for a start it has to be an upper bound.
>
> *Interviewer*: Right so what does that mean?
>
> *Lucy*: An upper bound for a set S, if you take any element of S to be a, say, and for all a you can find, say the upper bound was k, for all a, k will be greater than or equal to a for any number in that set.
>
> *Interviewer*: So that's the definition for k being an upper bound.
>
> *Lucy*: … and the least upper bound is also an upper bound but it's the least of all the upper bounds so l has to be less than or equal to k for all k greater than or equal to a.

In the fourth interview she is very confident expressing the definition verbally:

> Well it's got to be an upper bound itself and it's got to be the least of all the upper bounds.

But even in the fifth interview, when asked to write down the definition of the least upper bound of a non-empty set $S \subset \mathbb{R}$, she wrote:

> $\forall\, s \in S,\ s \leq \mu$ [saying "μ is an upper bound"]
>
> $\forall\, k \in \mathbb{R}$ s. t. $s \leq k$ and $\mu \leq k$,

After a discussion she modified the last part to "$\forall s \in S,\ s \leq k \Rightarrow \mu \leq k$." Although she had most of the component parts of the definition, she still needed to negotiate the details.

Sean's concentration on his earlier experiences

Sean continued to have difficulty with the concept of least upper bound, as well as the definition, throughout the course.

> *Sean (Interview 2)*: [you get the supremum by] looking at all the elements of the set to find out which is the greatest and choosing that number. I always have trouble remembering whether the supremum has to be in the set.
>
> *Interview 4*: It's the greatest number of … it's a number that's bigger than all the numbers in the set.
>
> *Interview 5*: The set {1, 2, 3} has upper bound 3. [Is 7 an upper bound?] No, it's not in the set.

In the second interview when asked for the least upper bound of the set S of real numbers x where $x<1$, he suggests "a very small number subtracted from one" or "nought point nine nine nine recurring", thereby maintaining his (erroneous) belief that the least upper bound is in the set. In the fifth interview he is quite articulate about his struggle:

> … when we have theorems in analysis lectures, stuff like supremums are just the basic workings; since I can only just understand these individually, one of these basic foundations, I can't look at all of them together and understand the theorem.

It is a classic case of cognitive overload. However, not only does he seem to have too many things to think about, the individual items not only lack the precision of operable definitions, they seem diffuse and difficult for him to grasp as manipulable mental entities.

Alex eventually learns the definition with apparent meaning

Alex is somewhat erratic in attending lectures so he does not get all the information from the course that he should. In interview 2 he explains an upper bound L for a set S, saying:

> *Alex*: there exists L such that L is greater than or equal to max S – what do you call it – the greatest number in S.
>
> *Interviewer*: Has S got a maximum number?
>
> *Alex*: Yes.

In the third interview he seems to become entangled with the completeness of the reals:

> [the least upper bound is] not necessarily in the set – well it depends – if your subset is a subset of the reals then it's going to be a real number, in which case it's going to be in the set but if you're talking about a subset of the rational numbers, it's not necessarily in it, but it's the lowest number that is an upper bound.

Yet, in the fifth interview he suddenly offers the full definition, writing:

> $l \leq s \; \forall \; s \in S$ [saying "l is an upper bound"]
>
> If u is an upper bound [pointing to the previous line, saying "satisfies this as well"] then $s \leq u$.

He explained that, since it arose several times in the interviews, he decided to *learn* it.

The Definition of Continuity

When the definition of continuity is given, the students have already met a succession of ideas including convergence of sequences and series. Lucy works at making sense of each new idea without always having the space to absorb earlier detail. For instance, when studying convergence of series she has no recollection of the definition of convergence of a sequence but, when asked for the definition of convergence of a series, she responds with a precise formal definition of Cauchy convergence of its partial sums. Although she tries to memorise the definition of continuity, it "would not stick", and she builds it up from its parts (sometimes by visualising the page of notes in her imagination). By the fifth interview she is fairly fluent with the definition of continuity and when asked to explain the definition of f being continuous on $[a, b]$ she constructs her own version in steps, writing the following down in order (1), (2), (3):

$$\underbrace{\forall x, x_0 \in [a,b]}_{(2)} \quad \underbrace{\forall \varepsilon > 0 \, \exists \delta > 0 \text{ s.t.}}_{(1)} \underbrace{|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon}_{(3)}.$$

Further conversation leads to this being modified as she explains:

> I write down everything and say, "no that's wrong", and then I work backwards.

Neither Sean nor Alex ever come to terms with the formal definition of continuity.

> *Sean (interview 4)*: It means the function at that point minus the function at some other point very near it is less than or equal to the epsilon.

> *Sean (interview 5)*: I can remember just about the definition of continuous but I tend not to use it and still think of continuity as drawing the graph without taking your pencil off the paper.
>
> *Alex (interview 5)*: It's a line you can draw without taking your finger off the board.

The Riemann Integral

The definition of Riemann integral $\int_a^b f$ of a (bounded) function f is introduced later in the course. The interval $[a,b]$ is partitioned into what (in the course) is called a *dissection*, D, consisting of sub-intervals $a = x_0 < x_1 < \ldots < x_n = b$. From the definition of least upper bound it is deduced that on any interval a bounded function has a supremum (least upper bound) and an infimum (greatest lower bound). Using the supremum and infimum of the function in each sub-interval to give an upper and lower rectangle allows the upper sum $U(f,D)$ and lower sum $L(f, D)$ to be computed, as the total area of upper and lower rectangles, respectively. Proving that $L(f, D_1) \leq U(f, D_2)$ for any dissections D_1, D_2 shows that the set of lower sums has a least upper bound L and the set of upper sums has a greatest lower bound U. If $L=U$, then the Riemann integral is defined to be $\int_a^b f = L = U$. If f is continuous, then it is proved that $L=U$ so the Riemann integral exists.

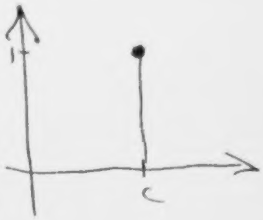
When the three students meet this sequence of theory, Lucy is already able to speak about both the definitions of least upper bound and continuous function and write them down (though sometimes with errors), Alex can write down one but not the other and Sean can write neither. So how do they cope with the definition of Riemann integral?

Lucy is able to discuss it intelligently, though needing occasional assistance. For instance, she says that a dissection of an interval consists of "lots of little bits" and, after a suitable notation is suggested, is able to describe both upper and lower sums, and upper and lower integrals as "the inf and sup of the upper and lower sums." She has a broad grasp of the overall framework including most definitions and statements of theorems, such as "a continuous function on a closed interval is bounded and attains its bounds", although she is still "trying to understand it at the moment but I haven't got it quite." However, she confidently explains why the function $f(x) = \begin{cases} 1, & x = c \\ 0, & x \neq c \end{cases}$ has a Riemann integral in terms of a zero lower sum and an arbitrarily small positive upper sum.

In the final interview, Sean claims no knowledge of the Riemann integral, saying:

> I can only say integrate using A level knowledge – I don't know how you'd do it using the theorem.

After the interviewer has talked through the definition of the Riemann integral, Sean is asked about the integral of $f(x) = \begin{cases} 1, & x = c \\ 0, & x \neq c \end{cases}$. He draws a graph with a vertical line from $(c, 0)$ to $(c, 1)$ and turns to his notion of integration as the area under the graph explaining:



> there's no way you can determine the thickness of that line and when you let things tend to zero you can't account for that, so there's no way you can work out the area in that bit.

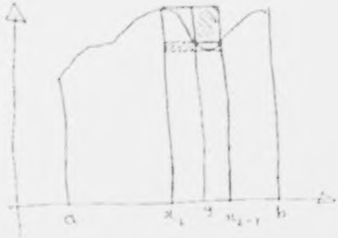
When the interviewer offers a demonstration that this function is integrable in the Riemann sense by considering upper and lower sums, Sean responds:

> Ha – I can't see any flaws in your logic, but I don't like it. … Because if you try to define an area you say it's something contained in this thing. … If you had a hole where there was nothing you could have great trouble finding the area at that point.

He has no operable definitions and no meaningful concept of proof from definitions. Instead he attempts to translate everything into his own intuitive terms and ends up with a diverse system of ideas that is just too unwieldy to make any sense.

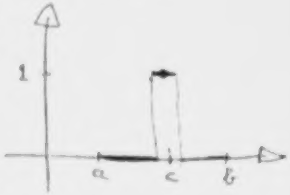
In contrast, Alex produces a most fluent account of the definition of Riemann integral. In particular he remembers and uses accurately the notations $U(f, D)$ and $L(f, D)$ introduced for the upper and lower sums for a function f and dissection D, and U, L for the supremum and infimum of the upper and lower sums. He is asked how to define the integral, and replies

> *Alex*: Well the integral is – oh dear – the integral is when that equals that [pointing to the symbols L and U] it's that becomes – oh – it's the supremum of the lower bounds and the infimum of the lower [sic] bounds become -- no wait – he did write the integral sign with a lower thing for the lower integral – or something – which was the supremum of this ([pointing to L].
>
> *Interviewer* (aside): The supremum of all the Ls – yes, fine.
>
> *Alex* (pressing on): … and you get your upper integral which is the infimum of your upper thing – so what happens is – like – the integral is when these two equal each other so you've got to like take more divisions, cos if you've got another division in there then – it's that thing where you've got L – if you've got D_1 which is a number of divisions here and you've got D_2 which just contains another one, then you've got $L(D_1) \le L(D_2) \le U(D_2) \le U(D_1)$.
>
> That's because you've now got another little partition up there (points to the diagram) so that plus – it's like, this isn't – this part and that part are now your upper, so you've lost some – whereas your lower has gone to there so you've actually gained some and so it goes closer and closer together, as you get more things until those are equal.



The status of the various components of Alex's discussion is intriguing. In the absence of an operable definition for continuity, he makes no formal mention of it or of any other property that would cause the upper and lower sums to be arbitrarily close. Using the diagram he sees the infimum and supremum as points on a ("continuous?") graph and imagines them becoming close as the intervals decrease in width. He *does* have a definition of least upper bound which is more formally operable and, by showing every upper sum exceeds every lower sum, he is clearly showing that the lower sums are bounded above and have a least upper bound, with the corresponding greatest lower bound for the lower sums. However, on occasions he seems to imagine variable upper and lower sums getting "closer and closer together." In this sense he is working with the *process* of moving towards a limiting value rather than using the definition of the limit concept.

When faced with the integral of $f(x)=\begin{cases}1, & x=c\\ 0, & x\neq c\end{cases}$, he draws a picture and focuses on the limiting behaviour of upper and lower sum as the interval width tends to zero.



Interviewer: Now if you took your partition – your lower sum is always going to be zero and your upper sum is only going to differ from zero in the interval or two intervals that contains that. If you take these very small …

Alex: Yeah – if you're timesing that distance there by this height, it will disappear.

Interviewer: So do you think that's integrable or not?

Alex: That's not going to be integrable – no, that's just going to give us zero isn't it?

Interviewer: Is the upper sum going to be zero?

Alex: The upper sum's going to be – that is going to be integrable in that case because the lower sum is always going to be zero, and the upper sum is going to go to zero as that gets smaller.

This extract shows interesting uses of the present and future tense. The interviewer asks "do you think that's integrable or not?", intimating his view of the state of the function. The response "that's not going to be integrable" suggests a sense of process, perhaps relating more to the process of allowing the interval width to tend to zero.

Discussion

In defining the notion of "formally operable definition" we hoped to have a construct which enabled us to see instances of the successful use of definitions in theorems to build a systematic formal theory. Of the three students here, Sean claims to "only just understand the 'basic foundations' individually" and "can't look at them all together." Definitions for him are not operable and the ideas are so diffuse that he cannot comprehend them. He does not understand the notion of a definition being used to prove anything. Alex, a more sophisticated but erratic performer, has no formally operable definition of continuity, but he is eventually able to formulate the notion of least upper bound in an effective manner and use it operably in the definition of Riemann integral even though other ideas (such as continuity) are not formally operable. Lucy is more effective, but the definitions are not committed to memory, rather constructed and re-constructed in a struggle for coherent meaning. She "writes down what she knows" then says "that's not right" and "works backwards". Throughout the interviews, sometimes with the assistance of an ongoing dialogue, she is able to build impressive links between the materials, even when there are a significant number of gaps in them. During the lecture course, she often continues to have difficulty with proofs long after she has been presented with them. In other words, the "operability" of the definitions are for her an ongoing struggle. There are at least two distinct components of operability, the giving of meaning to the definition itself, not only through examples, but through the development of a range of strategies for its use in different theorems.

A telling difference between the students is the manner in which particular verbal expressions may be helpful or unhelpful in moving towards the formal definition from the very outset. Lucy's initial conversations about the least upper bound are well-targeted from the outset whereas Sean states that "it's the biggest element in the set." Alex has some aspects of both. When he interiorises the definition of least upper bound, it gives him the impetus to use the focused idea in the subtle construction of the notion of Riemann integral – a task which Sean is not even able to start.

We hypothesise that there is an important principle underlying this observation which is more than just the use of particular terms. If a student is focusing mainly on the essential properties in the definition then, in meeting new examples, there is the possibility of focusing *only* on these essentials, thus greatly reducing the cognitive strain. A more diffuse view of the possibilities means that successive examples may have a variety of extra detail that can cloud the issue. The former approach has prior focus on the "intersection" of the properties of the examples, the latter must sort out the important essentials from the "union" of the examples with their subtle irrelevancies that can lead to cognitive overload. This research has considered a mathematics lecture course "as it is". New research is required to see if an explicit focus on the use of properties in a definition can lead to a better comprehension of systematic proof. It is not just a matter of how quickly the ideas are encountered in a mathematics course, but of the individual's capacity to focus on the role of the ideas in the overall theory.

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BEYOND 'STREET' MATHEMATICS: THE CHALLENGE OF SITUATED COGNITION

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In this paper the perspective of situated cognition is used to analyse the results of three year case studies of two schools. Students who were enculturated into a system of active knowledge-use are shown to be more effective than those who learned through transfer-based models of teaching. It is argued that the social, relational nature of knowledge that the students' developed is inconsistent with notions of 'street' and 'school' mathematics.

In the not so distant past mathematics was simply thought of as mathematics. A stable, elegant, abstract subject, to be learned, transferred and applied. As we move into the 21st Century ideas about knowledge are changing. In the same way as human intelligence, once regarded as a unitary possession, is now thought of in terms of multiplicities (Gardner, 1993), acceptance of the different *forms* of mathematics, such as 'school' and 'street' (Nunes, Schliemann & Carraher,1993) is widespread. More fundamentally, Lave and others in the field of situated cognition have suggested that all knowledge is situated and that human cognition is structured by social situations (1988, 1991, 1993). Thus, when knowledge is brought to bear upon a situation it is always a product of the people, their activities, their interests and goals and the ways that these relate to the situation they are in. It is not surprising that settings of the 'school' and the 'street' generate different forms of knowledge in this perspective, as all knowledge is thought to be shaped by the moment in which it is communicated.

My aim in conducting case studies of two schools was to explore these different notions of situated cognition, 'street' and 'school' mathematics, within schools. There isn't the space to report upon these case studies in any depth, but details of the two schools and the general issues that emerged from them are provided elsewhere (Boaler,1997).

The Research Study

In order to investigate the mathematical knowledge and understanding that students develop over time I performed longitudinal, ethnographic, case studies of two schools. These included 100, one hour lesson observations in each school, questionnaires, interviews and a variety of assessments. I followed two 'year groups' of students, about 300 in all, from the beginning of year 9 (age 13) to the end of year 11 (age 16). The students in the two schools were matched at the start of the research in terms of sex, social class, gender and mathematical attainment. Prior to the beginning of year 9 the students had experienced the same teaching approaches — working through small booklets designed to teach the students mathematics, with no teacher input from the front. At the beginning of year 9 one group of students moved to a traditional, textbook approach whilst the other group moved to an open-ended, problem-solving approach. I will now summarise the data that I collected from the two schools over the next three years.

Amber Hill School

'Amber Hill' is a large, mixed, comprehensive school, run by an 'authoritarian' (Ball, 1987) headteacher. The school is disciplined and controlled, there are numerous school rules that students follow and the corridors and classrooms are quiet and calm. Mathematics lessons in Amber Hill School are typical of those in many of the UK's secondary schools. In years 7 and 8 the students work through individualised booklets in mixed ability groups. In years 9, 10 and 11 students are taught in 'ability' groups, teachers demonstrate mathematical methods for approximately 15 minutes at the start of lessons and then give students questions to work through from their textbooks.

In lesson observations I was repeatedly impressed by the motivation of the Amber Hill students who would work through their textbook exercises without complaint or disruption. In a small quantitative assessment of their time on task I recorded the number of students who were working ten minutes into, half way through and ten minutes before the end of each lesson. Observing 158 students over an eight lesson period, over 90% of the students appeared to be on task at each of these times. Despite the students' apparent motivation however, there were many indications that students found mathematics lessons boring and tedious. Students demonstrated a marked degree of disinterest and uninvolvement in lessons, demonstrated by rows of students quietly copying down methods without any apparent desire to challenge or think about mathematics. In response to a questionnaire item asking students to write about aspects of lessons which they disliked (n = 160), 48% of students complained about their lack of practical or activity based experience and 31% criticised the *similarity* of the school's approach.

As a result of approximately 100 lesson observations at Amber Hill, I classified a variety of behaviours which seemed to characterise the students' approach to mathematics. One of these, I termed cue-based thinking (Boaler,1997; Schoenfeld, 1985). At many times during lessons I witnessed students basing their mathematical thinking upon what they thought was expected of them, rather than the mathematics within a question. Usually this expectation was based upon a structural aspect of their text books —what they thought should be demanded of them at a certain stage — or the context within a question. If a question seemed inappropriately easy or difficult, if it required some non-mathematical thought or if it required an operation other than the one they had *just* learned about, many students would stop working. The students did not interpret their work using mathematical sense-making or understanding, they searched for cues which prompted a familiar procedure used in a 'similar' situation. I asked them in a questionnaire (n=160) to say which they believed to be more important when approaching a problem: remembering similar work done before or thinking hard about the work in hand; 64% of students said that remembering similar work done before was *more important* than thinking hard about the current situation.

A number of different sources also showed that students at Amber Hill had abandoned trying to interpret situations mathematically and viewed mathematics as a series of 'rules, sums and equations' that needed to be learned (Boaler,1997). The Amber Hill students' mathematical experience was structured and orderly; students completed a lot of work and learned a lot of different methods. For many this experience was also characterised by a lack of meaning, a predominance of anxiety about understanding, belief in the need to learn set rules and a lack of critical thought.

Phoenix Park School

'Phoenix Park' school is different from Amber Hill in many respects. It is a small, age 13-18, upper school, well known for its tradition of progressive education and its concern for equal opportunities and special educational needs. In mathematics lessons the students work on open-ended projects, in mixed-ability groups. A strong theme which is important to the Phoenix Park approach is independence. In mathematics lessons the students are given starting points, for example, "The volume of a shape is 216: what can it be?" or "A farmer has 36 gates, what shape and size of fences can she build?". Students are then expected to work on these ideas for approximately three weeks, developing their own thoughts, collaborating with partners, taking the work in interesting directions and *using* mathematics. If students need to learn about a new mathematical idea or procedure, teachers explain it to them within the context of their project. For example, in the project on 36 fences some students wanted to work out the area of a 36-sided shape, so the teacher taught them how to use tangent ratios.

At Phoenix Park there was no apparent structure to lessons and, in contrast to Amber Hill, very little control or order. If students wanted to they could take their work to other rooms and work unsupervised, as they were expected to be responsible for their own learning. In lessons it was common for approximately one-third of students to be wandering around, off task. A study of the number of students working ten minutes into, half way through and ten minutes before the end of each lesson showed that approximately 60% of students were on task at these times, from an observation of 230 students over an eleven-lesson period. However, when the students were working, they needed to be thinking. In response to the questionnaire item concerning memory or thought, 65% of Phoenix Park students prioritised thought, compared with 36% of Amber Hill students, as noted above. When I asked students in interviews to describe their lessons to me, they talked about the relaxed atmosphere at Phoenix Park, the emphasis on understanding, the choice they experienced and the need to explain methods.

In order to monitor the development of the students' understanding in the two schools I used a variety of assessments over the 3 years which produced the following results:

- In short written tests set in different contexts, that were similar in style to the Amber Hill students' textbook work, there were no significant differences in the attainment of the cohort of students at the two schools at the end of years 9 or 10 (ages 13 - 15).

- In an 'applied' task given to two groups of students at each school at the end of year 9, Phoenix Park students attained significantly higher grades than Amber Hill students ($\chi^2 = 4.44$, d.f. = 1, $n = 104$, $p < 0.05$). At the end of year 10, a second applied task was given to 4 groups of students at each school. By this time the differences between the groups were even more marked. Phoenix Park students again attained significantly higher grades than Amber Hill students on all aspects of the activity ($\chi^2 = 17.46$, d.f. = 3, $n = 188$, $p < 0.001$). In tests that were designed to cover the same areas of mathematics as the tasks, given to students at the same time, there were no significant differences between the two schools. The Amber Hill students showed that they could use mathematics in tests, but many could not use the same areas of mathematics in applied situations. The Phoenix Park students were equally capable in both.

- In the national school leaving examination, the GCSE, which was similar in style to the Amber Hill students' textbook work, the Phoenix Park students attained significantly higher grades. Eighty-eight per cent of the Phoenix Park cohort passed the examination compared with 71% of the Amber Hill cohort, ($\chi^2 = 22.22$, d.f. = 1, $n = 290$, $p < 0.001$). My analysis of the types of questions that students answered correctly showed that the Amber Hill students solved approximately half as many 'conceptual' questions as 'procedural'; at Phoenix Park students solved equal proportions of each type (Boaler, 1997).

In years 10 and 11 I interviewed 40 students from both schools and asked them to think of situations when they used mathematics outside of school. The Amber Hill students *all* said that they abandoned school mathematics and used their own methods. This was because the students could not see any connection between the mathematics they learned in school and the demands of their lives. Over three-quarters of the Phoenix Park students interviewed (n = 36) said that they used their school learned methods in situations outside school. The Phoenix Park students reported that they did not perceive any real differences between the mathematics of school and the 'real world'. Thus, although the Amber Hill students spent more time on task (Peterson & Swing, 1982) in lessons and completed a lot of textbook work, whilst the Phoenix Park students spent a large proportion of their lessons wandering about the room or chatting, it was this latter group of students who were more able to use mathematics in a range of settings.

Discussion and Conclusion

The students at Amber Hill were consistent in their mathematical behaviour. They were motivated and hard working and they tried to learn all the procedures that were presented to them in class. However, in applied settings and examination questions many of them found that they were unable to use the mathematical procedures they had learned. At Phoenix Park the students showed that they were mathematically more competent in a range of situations. This appeared to derive from:

- a willingness and ability to perceive and interpret different situations and develop meaning from them (Gibson, 1986) and in relation to them (Lave, 1993, 1996)
- a sufficient understanding of different procedures to allow appropriate procedures to be drawn upon (Whitehead, 1962)
- a mathematical confidence and understanding that led students to adapt and change procedures to fit the demands of new situations (see Boaler, 1997).

The students at Phoenix Park had not been taught *about* mathematical procedures, they had been 'apprenticed' into mathematical use. When they encountered mathematical problems they did not try and 'transfer' set pieces of mathematics as the Amber Hill students did, they reflected upon their past experiences and changed the methods they knew to fit the situations they were in. The students themselves developed dynamic, relational views about knowledge-use which enhanced their mathematical success:

> JB: Is there a lot to remember in maths?
>
> S: There's a lot to learn, but then you need to know how to understand it and once you can do that, you can learn a lot.
>
> P: It's not sort of learning is it?, it's learning how to do things.
>
> (Philip & Simon, Phoenix Park, year 11)

> JB: How long do you think you can remember work after you've done it?
>
> G: Well I have an idea a long time after and I could probably go on from that, I wouldn't remember exactly how I done it, but I'd have an idea what to do.

> (Gary, Phoenix Park year 11)

In the first of these extracts Philip and Simon concur with Lave's claim (1996) that notions of knowing should be replaced with notions of doing, in order to acknowledge the relational nature of cognition in practice, as illustrated by the distinction drawn out by Philip: 'It's not sort of learning is it?, it's learning how to do things'. This comment also highlights the difference between the Amber Hill and Phoenix Park approaches. At Amber Hill teachers tried to give the students knowledge, at Phoenix Park the students 'learned how to do things'. This distinction led to differences in the mathematical beliefs of the students — at Amber Hill the students thought that they needed to remember a vast number of rules and procedures; at Phoenix Park the students thought that mathematical use involved reflection, thought and adaptation. Gary's comment is also important because he appears to suggest, quite explicitly, that he does not 'transfer' pieces of knowledge, rather, he creates new ideas in relation to the situations he is in. Gary's comment supports a relational view of knowing, because he dismissed the view that knowledge existed in his head ('I wouldn't remember exactly how I done it') and stated that his knowledge would only be *informed* by previously held ideas, he would 'go on from that' and form ideas of what he had to do in different situations.

A common theme running through the Phoenix Park students' reflections was the idea of change and adaptation:

> L: Yeah when we did percentages and that, we sort of worked them out as though we were out of school using them.
>
> V: And most of the activities we did you could use.
>
> L: Yeah most of the activities you'd use — not the actual same things as the activities, but things you could use them in.
>
> L: Sometimes I know I have changed methods to make it easier for me — if you find it easier the way you learned it then you keep the same, whatever's easiest. (Vicky & Lindsey, PP, year 11, JC)

These students support a situated view of learning (Lave, 1993), because they describe the way in which they developed meaning in interaction with different settings. Lindsey said that she would use mathematics 'not the actual same things as the activities, but things you could use them in', she would adapt and transform what she had learned to fit new situations. Later in the interview she said:

> L: Well if you find a rule or a method, you try and adapt it to other things, when we found this rule that worked with the circles we started to work out the percentages and then adapted it, so we just took it further and took different steps and tried to adapt it to new situations. (Lindsey, PP, year 11)

The analysis offered by Lindsey in this extract is very important, for it was this willingness to adapt and change methods to fit new situations which seemed to underlie the students' confidence in their use of mathematics in 'real world' situations. Indeed many of the students' descriptions showed that they had learned mathematics in a way that transcended the boundaries (Lave, 1996) that generally exist between the classroom and the 'real world'.

The results of this study lend support to some of the emerging ideas within situated cognition. For example, they suggest that attempts to impart knowledge to students, as characterised by dominant models of mathematics teaching in the UK, may be less helpful than classroom environments in which students are enculturated and apprenticed into a system of knowing, thinking and doing. The data also show the importance of students' beliefs, goals and interpretations, to their degree of mathematical effectiveness. These important features of their practice, central to theories of situated cognition, go unnoticed in solely cognitive interpretations of performance (Greeno, 1997).

Ideas of 'street' and 'school' mathematics have been allied to theories of situated cognition for a number of years yet they are, in many ways, diametrically opposed to each other. It became clear in this study that the Phoenix Park students did not develop knowledge forms that were as distinct or stable as those suggested by the labels of 'street' or 'school'. The students showed that the mathematics they used, in the street, the school or elsewhere, was developed for that particular moment in a process of reflection, adaptation and communication. A major claim of situated cognition is that knowledge is not only influenced by, but structured by the social setting. Such settings will also be variable even within the confines of the 'street' or the 'school'.

In Bishop's review of a 1991 book by Saxe, which considered the question of in-and-out-of-school learning, he writes that 'I was very interested to see not the total separation in the children's minds that I had expected, but the gradual interweaving of the two sets of cognitive practices' (quoted in Kieran, 1994). I would go further and challenge the very idea of 'sets' of practices. Kress (1996) has levelled a similar argument at the plurality of literacies that has been proposed, saying that: 'This paradox only exists if we assume that language is autonomous, unaffected by the social and therefore stable. If we assume that language is dynamic because it is constantly being remade by its users in response to the demands of their social environments, we do not then have a need to invent a plurality of literacies.' (Kress, 1996. p115). The data from students at Phoenix Park suggest that mathematical use has much to share with language-use in this respect, more perhaps than many would acknowledge. This is not to say that ideas of 'school' and 'street' mathematics have not been useful — they have, not least because they have helped to dispel claims of a single, autonomous knowledge (Street, 1997). But such notions were limited in their applicability for the students at Phoenix Park because the dynamic, 'relational' (Lave, 1996) nature of the knowledge they developed was simply unexplained by such stable models of knowledge-use.

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THE "VOICES AND ECHOES GAME" AND THE INTERIORIZATION OF CRUCIAL ASPECTS OF THEORETICAL KNOWLEDGE IN A VYGOTSKIAN PERSPECTIVE: ONGOING RESEARCH

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This report presents some new findings about the "voices and echoes game"(VEG), an innovative educational methodology conceived in a Vygotskian perspective and aimed at approaching theoretical knowledge, overcoming the intrinsic limitations of both traditional and constructivistic approaches. Based on some improvements in the theoretical framework of the VEG, new teaching experiments were performed. Analysis of student behaviour allowed investigation of some individual and social cognitive processes underlying the VEG, especially concerning the interiorization of some aspects of theoretical knowledge.

1. Introduction

In Boero et al (1997) an innovative educational methodology was presented: the "voices and echoes game"(VEG). Based on Bachtin's construct of "voice", the VEG was conceived with the aim of mediating (in a Vygotskian perspective) theoretical knowledge, overcoming the intrinsic limitations of both traditional and constructivistic approaches.

The aim of this paper is to present our recent findings about the potentialities of the VEG and the individual and social cognitive processes underlying it. Preceding results (see Boero et al. 1997, Discussion) and further reflexions about the nature of theoretical knowledge (see 3.1.) have allowed us to plan new teaching experiments and better analyse student behaviour. These experiments have involved different grades (from VII grade to university entrance mathematics courses) and different subjects (from mathematical modelling of falling bodies and heredity, to mathematical analysis). Analysis of some episodes taken from the performed experiments, such as those reported in Section 3, opened the way to a better understanding of the cognitive processes underlying the VEG, especially concerning the interiorization of crucial aspects of theoretical knowledge (see 4.).

This paper can be considered as a mid-term report about our research project. Further theoretical refinements and experiments (concerning new mathematical topics and tasks) are needed to increase present knowledge about the potentialities of VEG and the cognitive processes underlying it (see 5.).

2. Recalling and Refining the Theoretical Framework

The purpose of this section is to provide essential background information, as well as (in Subsection 3.1.) some improvements in the theoretical framework.

What is the VEG? Some verbal and non-verbal expressions (especially those produced by scientists of the past - but also contemporary expressions) represent in a dense and communicative way important leaps in the evolution of mathematics and science. Each of these expressions conveys a content, an organization of the discourse and the cultural horizon of the historical leap. Referring to Bachtin (1968) and Wertsch (1991), we called these expressions *'voices'*. Performing suitable tasks proposed by the teacher, the student may try to make connections between the voice and his/her own interpretations, conceptions, experiences and *personal senses* (Leont'ev, 1978), and produce an *'echo'*, i.e. a link with the voice made explicit through a discourse. The 'echo' was an original idea intended to develop our new educational methodology. What we have called the *VEG* is a particular educational situation aimed at activating students to produce echoes through specific tasks: *"How might.... have interpreted the fact that...?"*, or *"Through what experiences might ... have supported his hypothesis?"*; or: *"What analogies and differences can you find between what your classmate said and what you read...?"*, etc. The echoes produced may become objects for classroom discussion.

Students' echoes: students may produce echoes of different types (depending on tasks and personal adaptation to them). We distinguished between *individual echoes* and *collective echoes* (these are produced during a classroom discussion which may start from an external voice or some of the individual echoes selected by the teacher as voices). In Boero et al. (1997), ***individual echoes*** were classified. In this report we will focus particularly on *resonances*, the situations of greatest interest. In this case the student appropriates the voice as a way of reconsidering and representing his/her experience; the distinctive sign of this situation is the ability to change linguistic register or level by seeking to select and go deep into pertinent elements *('deepening')*, and finding examples, situations, etc. which actualize and multiply the voice appropriately *('multiplication')*. The echoes which develop at the **collective** level may consist of series of individual echoes of the voice at the origin of discussion *('source voice')* with a high level of connection between successive echoes. In particular, both the examples related to the 'source voice' *(multiplication)* and the expressions and expressive registers *(deepening)* may undergo rapid and intensive enrichment. We called this phenomenon *'multiple echo'*.

What is the aim of the VEG? Our general, *initial* hypothesis on this issue was that the VEG might allow the students' cultural horizon to embrace some elements (counter-intuitive conceptions, methods far beyond students' experience, specialized kinds of organization of the scientific discourse) which are difficult to construct in a constructivist approach to theoretical knowledge and difficult to mediate through a traditional approach. The need to exploit the potentialities that emerged in the first series of teaching experiments forced us to try to *characterize better the elements of theoretical knowledge* to be mediated, in order to better organize (through appropriate tasks) and analyse their interiorization by students.

2. 1. More about Theoretical Knowledge

In seeking to refine our framework about theoretical knowledge, we have found three different sources of inspiration. Although they belong to different cultural domains and orientations, they seem to offer coherent and useful hints about peculiar characteristics of theoretical knowledge.

The seminal work of Vygotskij (1990, chap. VI) suggests that: theoretical knowledge is systematic and coherent; it allows production of judgements (predictions, validations...) about the experience through intentional reasoning based on highly organized and culturally rooted linguistic patterns; it organizes the experience (both material and intellectual) in connection with a cultural tradition; it needs a particular mediation to be trasmitted to new generations.

Wittgenstein's theoretical costruct of 'language games' (Wittgenstein, 1953) can be exploited to describe how the potentialities of language (particularly the possibility of defining, of proving, of making the rules of inference explicit and, in general, of eliciting and discussing peculiar characteristics of a theory) allow theories to be constructed, described and discussed. Wittgenstein's analysis of common knowledge (1969) suggests that it offers the grounding and basic grammar for culture (and 'certainty') at any level, but it reduces to a set of (possibly incoherent) pragmatic tools if not systematized by a theoretical discourse.

Sfard's recent investigation (Sfard, 1997) suggests that *"the discourse of mathematics may be viewed as an autopoietic system [...] which is continuously self-producing. According to this conception, the discourse and mathematical objects are mutually constitutive and are in a constant dialectic process of co-emergence"*.

Taking into account these references, we may try to point out some particular characteristics of theoretical knowledge in mathematics, by considering both the *processes of theory production* (especially as concerns the role of language) and the *peculiarities of the produced theories:*

* theoretical knowledge is organized according to explicit *methodological requirements* (like coherence, systematicity, etc.), which offer important (although not exhaustive) guidelines for constructing and evaluating theories;

* definitions and proofs are key steps in the progressive extensions of a theory. They are produced through *thinking strategies* (general, like proving by contradiction; or particular, like 'epsilon-delta reasoning' in mathematical analysis) which exploit the potentialities of language and belong to cultural tradition;

* the *speech genre* of the language used to build up and communicate theoretical knowledge has specific language keys for a theory or a set of coordinated theories - for instance, the theory of limits and the theory of integration, in mathematical analysis. The speech genre belongs to cultural tradition;

* as a coherent and systematic organization of experience, theoretical knowledge vehiculates specific *'manners of viewing'* the 'objects' of a theory (in the field of mathematical modelling, we may consider deterministic or probabilistic modelling; in the field of geometry, the synthetic or analytic points of view; etc.)

We think that the approach to theoretical knowledge in a given mathematics domain must take these elements into account, with the aim of mediating them in suitable ways; indeed each of the listed peculiarities is beyond the reach of a purely constructive approach. The next section will show how the VEG can function as a learning environment where some of the elements listed above can be mediated through suitable tasks.

3. Mediating what? Some Examples from Recent Teaching Experiments

EXAMPLE A (VII grade)

This example concerns the *methodological requirements* of theoretical knowledge; it is taken from a new version (organized taking into account the content of 3.1.) of the "fall of bodies" teaching experiment reported in Garuti (1997). For further details about the new version, see Boero & Tizzani (1997).

The students met Aristotle's and Galileo's selected texts ("voices"), and were asked to echo them in tasks of different types: *"How might Aristotle have explained the fact that... ?"; "How might Galileo have opposed the idea that...?"* .

The following individual task was set: "*Galileo is convinced that Aristotle's theory* (proportionality between speed and weight of the falling body) *does not work and tries to prove that speed is proportional to height* (from which the body falls). *In order to better understand Galileo's position imagine dropping an object (not too restrained by air) from different heights, for instance one meter, three meters - first floor, nine meters - third floor). In your opinion, is the speed the same in the three cases? Why?*".

Michele wrote: "*...the body falling from the lowest height arrives first*"

Chiara wrote: "*If we drop a stone from one million km and another stone from one km, the latter will arrive earlier because it has to cover a shorter distance*".

Students were invited "*to discuss Michele's and Chiara's answers, taking into account Galileo's manner of reasoning*". The aim of this task was to help the students to overcome the Galileo's initial erroneous hypothesis *("speed is proportional to height")* through the methodological requirements of his own theory. During this discussion, Galileo's mistaken hypothesis was considered and opposed. Then the following episode was recorded (underlining indicates methodological aspects)

TEACHER: "*OK, then we may deduce that Galileo's hypothesis does not work. Chiara, according to you what relationship might exist between the speeds of the two stones?*"

CHIARA: "*If we drop a stone from a height of one meter at the same time when another stone starts from a height of two meters, the first stone will arrive before the second because it is nearer to the ground, but the second stone will have a higher speed - indeed, a speed which is twice the speed of the first*"

MICHELE: "*Then you do agree with Galileo!*"

CHIARA: "*Yes, but this agreement does not work for all the cases*"

MICHELE: "*Galileo said that a law must be valid for every case, even for two bodies falling from one meter or 10,000 meters*"

CHIARA: "*Indeed, I agree with Galileo only up ... to a certain point*"

MICHELE: "*Before telling us your hypothesis, you should have tested it for every case!*"

CHIARA: "*I was going to say it, Galileo criticizes Aristotle and then makes the same mistake, because his hypothesis does not work in all the cases.*"

EXAMPLES B) AND C) (Alessandria University Chemistry students)

These examples are taken from a long term teaching experiment concerning a one-semester intensive mathematics course intended to provide students with a common basic background concerning differential and integral calculus, analytic geometry, matrices and linear systems. The experiment consisted in the systematic proposition of some significant pieces of theory (crucial definitions, theorems, etc.) as "voices", alternated with tasks calling for production of conjectures and proofs requiring *similar ways of reasoning*, to make explicit the usage of *similar tools*, or to find counter-examples in *near* cases, etc. (these tasks were chosen in order to activate "resonance" echoes).

EXAMPLE B) concerns *thinking strategies*. Students were taught the "classical" proof that (given a real function f defined and twice derivable in an open interval containing 0, with the second derivative continuous in 0) if $f'(0)=0$ and

$f''(0)>0$, then the function f is concave upwards in $x=0$, that is its graphic is over the tangent in 0 in some neighbourhood of 0; this proof is based on analysis of the sign of the Lagrange remainder in Taylor's formula. As echo, they were individually invited to prove (using "*similar manners of reasoning*") that, under similar hypotheses, in the case of $f'(0)=0$, $f''(0)=0$ and $f'''(0)>0$, 0 is a point of inflection. Few students (5 out of 23) produced satisfactory answers. A short discussion followed, in which one student (Daniele) presented his reasoning and (following the teacher's sollicitation) made explicit the analogy/difference that he had discovered with the case n=2: "*also with three the sign of the third derivative remains positive in some neighbourhood of zero, due to the continuity of the third derivative; but unlike case two, the sign of x^3 changes from left to right of 0, and so the graphic of the function changes from below to over the horizontal tangent*".

In an evaluation test six weeks after these activities a specific task was set that concerned proving that a function such that $f(2)=1$, $f'(2)=0$, $f''(2)=0$, $f'''(2)=0$, $f^{(iv)}(2)=-3$ was concave downwards in 0 in the hypothesis of continuity of the fourth derivative. A satisfactory answer was provided by 15 students out of 23 (65%); 8 answers (33%) explicitly and autonomously reported how the "voice" concerning the case $f'(0)=0$, $f''(0)>0$ was recalled and exploited[*]. For instance, Giorgia wrote: "*In the case presented by the professor, the crucial element was to consider the Lagrange remainder in the Taylor's formula: $f''(c)x^2/2$ and prove that the sign of the remainder (which represents the difference between the tangent and the function) was positive in some neighbourhood of 0 due to the continuity of the second derivative (implying the permanence of sign) and the parity of the exponent of x; in the present case, the situation is similar, because the fourth derivative is continuous and the parity of four is the same as two.*"

EXAMPLE C) concerns the '*speech genre*' of a theory. Another task in the evaluation test considered above was to prove that, given a real function f defined and derivable for every real number, if $f(-1)=-1$, $f(1)=2$, $f(2)=0$, then there exists at least one point c_1 such that $f'(c_1)>0$ and one point c_2 such that $f'(c_2)<0$.

(Elisabetta): "*If f is derivable over **R**, then it is also continuous over **R**, consequently (by the Bolzano-Weiestrass theorem) between -1 and 1 there is at least one point c such that $f(c)=0$; thus by the Rolle theorem between c and 2 there exists at least one point d such that $f(d)=0$. We may assume that d is a relative maximum point because $f(1)=2$, consequently f is increasing in a neighbourhood before d and decreasing in a neighbourhood after d; as f increasing in a neighbourhood before d, in that neighbourhood there is at least one point c_1 such that $f(c_1)>0$ [....].*"

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[*] Three years before, the same evaluation test was done by 27 students enrolled in the same faculty of chemistry in the same university in the same period of the year by the same teacher. As is usually the case in Italian mathematics courses, those students had been taught the proof of both the case n=2 and the case n=3, and a generalization had been stated (without proof) by the teacher. The total time devoted to this topic was the same (about 75') as in the later course but the results were significantly different: only 9 students (33%) were able to produce the proof in the case n=4; only 2 (8%) explicitly and autonomously recognized the connection with the theorem presented in the case n=2. The difference in the results probably lies in the changes concerning both the activities performed on the specific topic and the general educational orientation of the whole course.

Note that although Elisabetta's proof is not valid, she uses the mathematical analysis language keys fluently *("if f is derivable, then..., thus..., consequently..." "there is at least one point c such that....",* etc*)*. Two thirds of the students produced proofs which were at the same level (as concerns the *'speech genre'* requirement) [1].

EXAMPLE D (VIII grade)

This example concerns the problem of the mediation of *manners of viewing*.

In this case, the experiment consisted in the exploitation of some pieces of Mendel's original paper (as 'voices') in order to introduce students to Mendel's theory through the VEG. A crucial moment during the experiment was the following 'resonance echo' task, concerning a plant (Mirabilis ialapa) with two varieties, one producing white flowers and the other red, which when crossed produce plants with pink flowers: *"Mendel was convinced that his ideas were right, and so, when he saw that in the second generation some plants still produced white flowers and others red flowers, he was able to predict the percentage of plants for every colour. What do you think were the percentages predicted by Mendel? How could Mendel interpret what happened with the first and the second generation?"*

Many students regressed to their spontaneous conceptions about heredity (mixing of father's and mother's characteristics, etc.); their quantitative prediction usually bore no trace of Mendel's point of view. Here is one example: *"The predicted percentage was 33,3%; [....] because adding the three percentages we must get 100%, and so I have divided 100 by 3, the number of colors, and the result 33,3% should be the percentage of plants for every color"*.

Only seven students (out of 27) adopted Mendel's point of view. This is a typical answer: *"If I were Mendel, I would explain what happened in this way: a) after crossing, the first plants bear pink flowers, because I think that in this case colors are determined by two dominant genes and we may assume that there was a mixture; b) if the two genes are dominant, the parents of the second generation must have WR and RW genes, consequently they produce plants with genes WW, WR, RW and RR, and so these plants bear (respectively) white, pink and red flowers [...]. on average, in the proportion 25%, 50% (WR and RW), 25%... "*.

This task proved to be very difficult; for an interpretation of the difficulties met by students, see Boero & Lladò, 1997.

4. The Interiorization Processes in the VEG

The reported examples may be analysed from different perspectives. In a Vygotskian perspective, the study of the processes of interiorization is one of the main questions; Davydov (1988, pages 33-34) explicitely refers to *"theoretical knowledge"* and *"theoretical thinking"* as crucial issues for interiorization. Taking into account the peculiar characteristics of theoretical knowledge listed in Section 2.1., we may try to understand how the students interiorized them through the VEG.

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[1] Concerning the importance of this aspect, it is interesting to note that Elisabetta was able to take active part in the discussion that followed. On the contrary, some other students who had produced more appropriate intuitions but had expressed them in a rough manner that had little in common with mathematical analysis language had great difficulties entering discussion in a productive way and found it especially hard to recognize the analogies between their thinking strategies and the strategies proposed by their fellow students.

The reported examples suggest that, in the case of the VEG, the students may follow different patterns of interiorization, *depending on the object of interiorization and on the educational context* (especially individual and collective taks). Indeed,

* Example **A** suggests that the *methodological requirements of a theory* may emerge at the level of shared consciousness in classroom discussions, through *multiple echo* phenomena (*multiplication* and *deepening* in the reported excerpt); opposing classmates' positions may induce students to select and make explicit methodological elements which were explicit in the "voice" but only potentially accessible to their attention (and indeed in the reported experiment practically no student quoted methodological elements in his/her initial individual echoes). As concerns the transfer problem for this kind of acquisition, we may add that in the same experiment we collected experimental evidence about the fact that the 'methodological consciousness' that emerged during classroom discussion may remain as a habitus in the individual performance of many students (in the shape of an inner discourse). In a Vygotskian perspective, it was as if outward questions were transformed into inward questions. As a representative example, we may quote the following excerpt from an individual production written two weeks after the discussion considered in Example A: "*I must check whether this assumption has no exceptions, because Galileo's position is that [....]*".

* Example **B** suggests that in the case of *thinking strategies*, some students may become conscious of the structural-cultural aspects of the strategy through the effort demanded by the echo task (by eliciting the aspects to be transferred to the case considered in the task). However, inquiry concerning the structural aspects, although needed to fulfil the task, in not spontaneously practiced by the majority of students, nor may it be induced through a more detailed specification of the task. Classroom discussion seems to be the appropriate environment for appraising, forcing and extending this kind of inquiry (by making it explicit in the teachers' sollicitations and *multiple echo* phenomena-*deepening* in the reported texts):the need to communicate and compare one's own solutions leads to an appropriate explicitation of the structural aspects which were already present in some students' echoes - see Daniele.

* Example **C** suggests that in the case of *speech genre*, interiorization can be a direct and rather spontaneous consequence of the practices of individual echoing and discussion about echoes. Indeed no explicit discussion was held in the classroom about elements of speech genre, and moreover we may remark that it is not easy to bring up "speech genre" as an object of a discussion. These remarks and classroom observations match what Vygotskij wrote about the potentialities of pure imitation in the zone of proximal development of students (Vygotskij, 1978, chap. VI) and the limited but certainly not irrelevant value of the learning of appropriate verbal expressions on the path to learning scientific concepts (Vygotskij, 1990, chap. VI).

* In the "Mendel's laws" teaching experiment (Example **D**) only about 25% of the students were spontaneously and unconsciously able to interiorize Mendel's *manner of viewing*. This result may be read in two different ways: as a potentiality of the VEG, or as a challenge about the possibility of improving the results. Boero & Lladò (1997) suggest that the results could probably be improved if this aspect of Mendel's theory were brought to surface as an explicit object of discussion by exploiting *both* pre-Mendel *and* Mendel's voices, the former being chosen in order to represent students' conceptions about heredity at the theoretical level. In other

words, the VEG should be integrated by tasks (and cultural contributions) which allow students to make explicit different mathematical systems of reasoning about heredity and compare them at the theoretical level.

5. Concluding Remarks

Only some elements, identified in Section 2.1. as peculiar characteristics of theoretical knowledge, were intentionally mediated through the experiments considered in this report. One issue for future investigation is the possibility of mediating other elements (possibly through different tasks concerning new topics: for instance, algebraic language).

As concerns the cognitive mechanisms underlying the VEG, the analysis performed in Section 5 suggests a complex perspective, where different variables must be taken into account: the alternation of individual activities and classroom discussions, the formulation of the tasks for these activities, etc. We may remark that the list of characteristics of theoretical knowledge outlined in Subsection 3.1. might be extended and should be improved as concerns the precision of the different points and the connections between them. These improvements might lead to further experiments and in-depth analyses of student behaviours, aiming at a better understanding of the conditions which allow the VEG to function productively and, possibly, a unified and simplified educational perspective for the approach to theoretical knowledge.

Finally, there lies the general problem of the space for the VEG in the activities performed in the classroom. Especially with younger students, it seem useful to alternate different kinds of tasks, limiting the VEG to a few, crucial topics for which a constructivist approach does not seem to be productive. Another option might be to alternate between different kinds of tasks when dealing with the same topic (production and explicitation of students' solutions and comparison with voices from mathematics and science). Furthermore, long-term experiments may highlight the potentialities and limits of these orientations (particularly as concerns the delicate problem of the didactic contract: the second orientation in particular demands frequent 'breaks', which may be confusing for students!).

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CHILDREN'S CONSTRUCTION OF INITIAL FRACTION CONCEPTS

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A constructivist teaching experiment was carried out with an entire class of year 4 children in order to explore and study their development of initial fraction ideas. Building on the naming conventions for fractions introduced in year 3, a series of mathematical games was used as a basis for the children's construction of fraction knowledge in a realistic social setting. Meanings constructed during the teaching experiment, revealed by videotapes of the learning activities, children's written records, and observations from the researchers and class teacher, were contrasted with those brought from earlier learning and their intuitive fraction notions. The study indicates how a base for an extended fraction concept can be linked to initial fraction ideas via activities situated in the child's world of play through negotiated, shared meanings.

Introduction

> *Although students may not construct fractions in out-of-school contexts, they do construct a wealth of informal knowledge on which we can base the teaching of fractions. Learning this informal knowledge is crucial but not sufficient. We must also learn how students actively construct fractions in school-based learning environments.* Steffe & Olive, 1991, p.24

Both the nature of what a fraction is and the means by which it might be represented to develop a broad understanding in children are problematic. Although some elementary knowledge of fractions as parts of things, specifically for halves and quarters, and a background of whole numbers is brought to the learning of fraction ideas in schools, children's difficulties in moving beyond these initial ideas are well documented (Murray, Olivier & Human, 1996). One reason is that fraction ideas are amongst the first abstracted mathematics met by a young learner, in that there is no natural context for fractions paralleling the experiences of counting or using groups of objects that underpin whole number learning (Booker, 1996a). Indeed, prior knowledge of whole numbers is often a hindrance to developing meaning for initial fraction ideas (Streefland, 1991) and, 'in spite of the fact that they often do get the right answer on school fraction tasks (eg "Shade one third'), their understandings of fractions may not be principled, but are based instead on remembered images' (Ball, 1993, p.175). Children's thinking about fractions is also made complex because of the variety of subconstructs that must be eventually be interwoven and because ideas of ratio and proportion are intrinsically concerned from the outset (Pitkethly & Hunting, 1996).

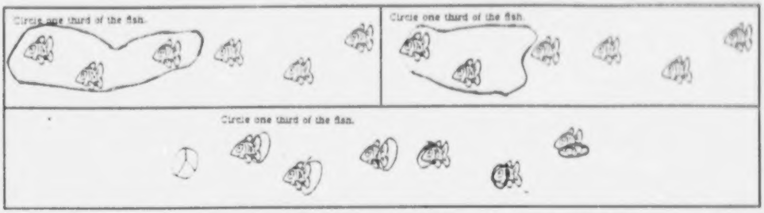
A full understanding of fraction ideas would seem to require exposure to numerous rational number concepts (Pitkethly & Hunting, 1996; Behr, Harel, Post & Lesh, 1993, 1992; Kierin, 1988). Kierin's (1988) model of mathematical-knowledge building suggests that this development proceeds through four levels, beginning with the basic knowledge acquired as a result of living in a particular environment, such as a recognition of parts and wholes and the names of the elementary fractions in everyday use. At the next level, this is broadened to an intuitive, schooled knowledge built from and related back to everyday experience, as in the case when initial number knowledge is extended to cope with the names of fractions in general. The third level includes the technical, symbolic language that involves the use of standard language, symbols, and algorithms while the fourth and final level consists of axiomatic knowledge of the system.

This study is concerned with the move from the level of naming fraction amounts to being able to use fraction symbols meaningfully in both naming and renaming contexts, including the initial ideas of equivalence, and thus to provide a basis for transforming whole number computational processes into ones for the various fraction forms. It is part of an ongoing investigation of the development of fraction understanding which, being constructivist in orientation, sets out to 'trace the development of fraction concepts in children' and reflect 'this natural development ... in constructing the curriculum' (Pitkethly & Hunting, 1996, p.32). The studies are set in a Catholic primary school which draws children from predominantly middle class backgrounds and attempts to build an enquiry based learning environment throughout all year levels. In the mathematics programs, the children have been introduced to ways of learning through games and other heuristic activities that encourage them to explore situations, look for patterns and relationships, make conjectures and discuss their interpretations and emerging ideas with others. Children are observed working in small groups on investigative tasks and problems or participating in mathematical games designed to bring out issues in the modelling, naming and renaming of fractions. These observations take the form of videotapes of small group and whole class activities and discussion, teacher and researcher observations and interviews with individuals or small groups.

Theoretical framework

Representing initial fraction ideas. The part-whole construct, based on partitioning either a continuous quantity or a set of discrete objects, appears to be fundamental to the development of fractions (Pitkethly & Hunting, 1996; Behr, et al, 1993; Kierin, 1988). The difficulty, though, is to select a representation or representations with which 'they can extend and develop their understandings of the ideas, as well as their capacity to reason with and about those ideas' (Ball, 1993, p.160). These need to be able to both facilitate the development of understandings and processes and allow students opportunities to explore and make conjectures concerning their emerging ideas and understandings. In this way, through making sense of the situations that arise, children can begin to appropriate these models and representations as their *own*.

This has usually been taken to mean that a notion of a 'unit fraction' is fundamental as it can be related to the part of the whole that names a fractional amount, and then built up to a more general fraction through a process of iteration (Pitkethly & Hunting, 1996; Steffe & Olive, 1993). Yet, historically, the notion of unit fraction is only one of the bases on which a successful fraction concept and scheme has arisen. While the Egyptian system of fractions can be interpreted as unit fraction in character, it is unique in this approach, and that evolved in other cultures, principally in the early Indian mathematics on which many of our procedures are based, took a compound fraction approach from the outset (Joseph, 1991). Perhaps, then, it is not surprising that a unit fraction basis for an introduction to initial fraction ideas has consistently proven difficult for children. For example, children commonly decide that 1/4 must be greater than 1/3 by generalising from their knowledge that 4 is greater than 3, or associate 1/3 with 3 objects or parts out of 6, rather than 2 objects, by relating the fraction name (third) to the number of objects within the collection or whole rather than to the subsets or parts that might be formed:


Circle one third of the fish.
Circle one third of the fish.
Circle one third of the fish.

12 of the 28 children in the study identified 1 third with three objects; only 5 children correctly circled two objects; 3 children circled 1 third of each object, 2 children circled 1 third of 1 object, while 4 children gave no response.

An alternative approach is to build from an initial conception of a fraction as a part of something to focus on one of something made up of parts in contrast to being used in the formation of larger whole numbers via place value ideas. These parts can then be used to consider compound fractions of the form 'so many parts out of the total number of parts' from the outset, and unit fractions would then simply occur as one case out of many for each partitioning of 1 one. Manipulatives as such would not be helpful, for when something representing 1 one is broken into smaller pieces it is as likely that this would be interpreted as several smaller ones rather than several parts of the original. Children constructing their fraction understanding need to experience this partitioning rather than be given materials which have already been formed into parts so that a model in which equal parts are completed and shaded seemed preferable to cutting something into equal pieces or reconstructing 1 one from ready-made parts. Thus, activities to reflect this partitioning process on the part of the individual learners were chosen, rather than using materials such as Cuisenaire rods which need to be interpreted from a fraction perspective, materials that could readily be broken apart such as plasticine or paper folding which most readily relates to a process of successive halving and thus leaves to one side more general notions of partitioning.

An emphasis on partitioning 1 one rather than taking parts of a whole also lays a foundation for fractions to be seen as numbers in an extended number system rather than as a way of using already known whole numbers, separated by a point

(3.46) or one on top of another ($\frac{3}{4}$). Just as 1 one was the base for all whole numbers (10 ones form 1 ten, 10 tens form 1 hundred and so on), children need to come to see that 1 one provides the basis for all fractions as well. The parts of 1 one give the initial fractions halves, thirds, fourths and so on, and in turn these generate all numbers by allowing for parts of 2 or more ones such as 1 and 3 fifths, 2.7, or 350%.

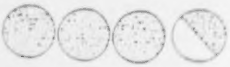
Instructional games as a basis for the social construction of meaning. Activities were designed to give situated meaning through the use of structured games and tasks as opposed to the focus on rule-like procedures that often dominate the learning of fractions. As Steffe and Wiegel (1994, p.117) suggest, 'playing in a mathematical context could serve in children's construction of a mathematical reality and as a source of their motivation to do mathematics'. Involvement in instructional games induces children to make sense of their ideas and the interpretations of others. The dialogue engaged in while playing facilitates the construction of mathematical knowledge, allowing the articulation and manipulation of each player's thinking. Such communication helps to extend a conceptual framework through a process of reflection and points to the central role of language, as it is the social interaction which gives rise to genuine mathematical issues. In turn, these problems engender an exchange of ideas with children striving to make sense of their mathematical activity and leads them to see mathematics as a social process of sense-making 'requiring the construction of consensual mathematical understandings (Booker, 1996b).

Method and procedure

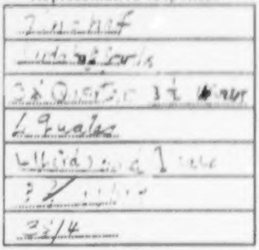
Classroom setting. In studying the development of fraction ideas, problematic activities were presented and children observed building up conceptual models, developing arguments, discussing their ideas and negotiating their understandings and interpretations. The teacher's role was to facilitate discussion and probe children's thinking through questions related to the children's emerging understanding of the activities and ideas that were being represented. At first, children were engaged in cognitive play activities to establish the enactment of basic conceptual thinking before the teacher intervened to transform this into mathematical activity which aided the construction of underlying fraction concepts (Steffe & Wiegel, 1994, p.118). Time was provided at the end of a set of activities for children to write up and reflect on what had occurred in individual diaries, and these, as well as issues that arose during the activities, formed the basis for whole class discussions bringing together a shared understanding of the developing mathematical ideas. In these discussions, the teacher acted as a facilitator of classroom discourse, posing questions and bringing forth consensual meanings for the underlying concepts as they emerged.

Initial assessment. Prior to the development of the new fraction ideas, an assessment of understanding retained from teaching and learning in the previous year was made. Although there was both informal knowledge from everyday experiences and a retention of many of the fundamental ideas developed in that year, there were also common misconceptions. Shading objects to show simple fractions or matching simple fraction names to shaded regions were carried out readily, but whether this really showed understanding had to be questioned given the range of responses to a situation involving several regions:

How much is shaded altogether?



Representative responses



3 n a haf
[illegible]
3 ½ Quarters 3 ½ [illegible]
4 Quales
4 thirds and 1 [illegible]
3 ½ [illegible]
3½/4

The sequence of learning activities[1] Following an analysis of the initial assessment data, the study began with some explicit review of the naming processes for fractions, building in the conventions that are necessary (Ball, 1993, p.165), in particular the notion of parts being equal-sized and the relationship of these new numbers to (but distinct from) ordinal number names. Following this, a game, *Colour me fractions*, was introduced in stages to build from a familiarity with naming rectangular regions to relating this to regions with other shapes, then building in the recording in all its complexities. This game followed a bingo format in that a shape relating to a particular fraction name was to be located and coloured just once on a game sheet shared by players to match the directions given by rolling two dice, one with digits from 1-6 (later 0-9), the other with fraction names halves, thirds, fourths, fifths, sixths, sevenths, eighths, ninths, tenths and twelfths. The players who first coloured each shape won the game, provided there was agreement that their responses matched the fraction concepts being developed.

The first playing board had rectangles with 3, 4, 5, 6, 8, 9, 10 and 12 equal parts so that from the outset some rolls of the dice would not generate a fraction to be coloured. The rule that if a particular shape already had some colouring then it was now 'out of play' raised acceptance of the fact that there may not be a shape corresponding to the dice (neither 3 sevenths nor 2 halves could be coloured) and provided a means to reflect on the notion that other rolls of the dice would also generate fractions that could not be coloured (6 fourths, 5 thirds, and so on). These possibilities were not broached with the class, but were left to arise as problems to be dealt with, conceptions to be negotiated on the route to building a robust fraction concept.

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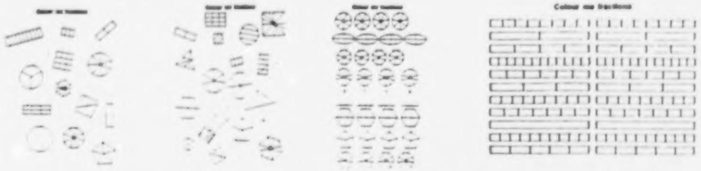
[1] Videotaped episodes and transcripts will be presented at the paper session. An overview is presented here as background to the video insights into the study and children' ways of thinking.

An extension to the playing board involved a variety of shapes and sizes, including examples where there were no partitioning marks raised the issue of children determining the particular partitioning themselves and thus creating an awareness of the variety of ways of viewing 1 one. Different shapes and sizes used to represent the same fraction were also designed to extend children's fraction concepts and give rise to argumentation of what was involved. In a final playing board in this section of the study, shapes with non-equal parts were included to assist in having the fraction concept to the fore rather than a procedure of simply finding and counting parts.

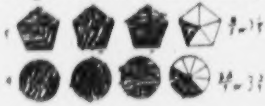
A further aspect of the sequence of games that had been constructed, is that by allowing some moves and denying others, situations arose where children wanted to resolve for themselves issues concerning the meaning of the fractions that they had rolled with the dice. In particular, when the question arose that you surely could have 3 halves (children provided examples using familiar objects such as apples or oranges), it became possible to extend the games by supplying game sheets with more than one copy of each shape:



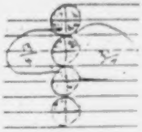
Game boards used in the study

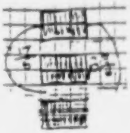
This in turn led to discussions about how the 'new fractions' might be recorded and a class discussion focussing on the link between earlier recording of fractions such as 7/8 and the models used to portray them led to a consideration of writing 13 eighths as both 13/8 and 1 and 5/8. At this point, there was a need to relate the class suggestions to the mathematical conventions that had arisen over time and this allowed students to construct the meaning for these recording conventions for themselves from the playing situations in which they were engaged. As a final set of games, versions of a 'tic-tac-toe' fraction comparison game were introduced (Booker, 1996a) and games using fraction representations in terms of discrete objects were devised based on conventional bingo and card games that encouraged the verbalisation of fraction names in both word and symbol form and the interpretation of continuous and discrete pictorial forms.

Results Children were able to construct for themselves strong and meaningful representations of quite complex fraction forms. These can be showed in the following succession of building from the initial fraction games:

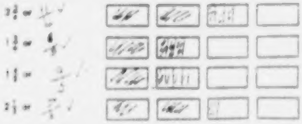


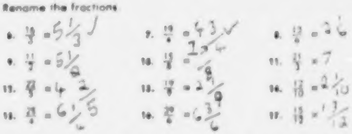
to representations they drew themselves in order to answers posed when the teacher rolled the dice and asked for two different ways of recording the resultant fractions





through homework assignments using both drawn representations and purely symbolic forms:




Rename the fractions

However, an ability to compare fractions in mixed number form did not build from this knowledge of naming and renaming fractions as might be expected, suggesting that a stronger notion of equivalence would be needed to be built in to their emerging connected pictorial, language and symbolic representations of fractions.

Circle the larger fraction:

Finding the larger fractions did not translate to mixed numbers where earlier whole number conventions continued to dominate

Conclusions and implications

The complex nature of thinking of fractions in a variety of ways, some consistent with and others divergent from, children's existing whole number knowledge has been brought out in their differing responses to fraction tasks. At the same time, the construction of meaningful conventions for renaming fractions has proven more successful than with the activities investigated in the earlier study (Booker, 1996a). The change from a solely part/whole conception of fraction to a more extendable part of 1 one concept appeared to reduce confusion in the renaming of mixed numbers and improper fractions. The building in of models that went beyond the rectangular region that built up an initial focus on a common notion of unit for fraction naming and comparison to different shapes and sizes of these representations, and then to collections of objects so that discrete fraction representations could be accommodated provided an extended base of intuitive ideas. This allowed more ready comparison of fractions through facilitating the move from symbol to language as a mediating notion. In this way, this research has supported Streefland's (1991) call for 'insightful reconstruction' of a system of fraction ideas. The next phase in this ongoing investigation will be to trace whether the conceptions that have been constructed are sufficiently robust to allow a meaningful notion of fraction as ratio to be intertwined, both in its own right and in the specific per cent form of fraction symbolism and representation.

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GRAPHING CALCULATORS AND REORGANIZATION OF THINKING: THE TRANSITION FROM FUNCTIONS TO DERIVATIVE [1]

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Abstract

In this paper we discuss three theories about how computers or graphing calculators can be related to cognition. We present some data from a mathematics classroom for biology majors at the State University of São Paulo at Rio Claro. In this course, functions and derivative are main concepts. The data presented regards an episode in which there is a transition from the work with functions to derivative. Finally, we discuss how the data presented is connected to the theories.

INTRODUCTION

Studies about derivative and ideas related to it (such as tangent lines) have emphasized students' misconceptions and common errors (Baldino, 1995; Scher, 1993; Artigue, 1991; Orton, 1983).In order to overcome some of these difficulties, the use of magnification to generate the idea of local straightness of differentiable functions often appears in the literature; Tall (1991) has proposed using a graphic approach using computers to do this magnification.

The effects of computers and graphing calculators on students' thinking on calculus and precalculus have been studied by different authors in teaching experiments with reduced number of students or in whole classrooms (Gómez & Fernández, 1997; Mesa, 1997; Villarreal, 1997; Mesa & Gómez, 1996; Borba & Confrey, 1996; Souza & Borba, 1995; Lawson, 1995; Hillel, Lee, Laborde & Linchevski, 1992).

Studies which stressed the (social) construction of mathematical knowledge in the classroom have been more common at the elementary level (Graves & Zack, 1996; Wood, 1996; Mousley & Sullivan, 1995). Although there are exceptions, little research has been done regarding this at high school or undergraduate level.

In this paper, we will emphasize the kind of discussion that emerges in a mathematical classroom when graphing calculators are used regularly. We will report on students' different understandings of a task which was meant to introduce the notion of derivative for first year biology majors. Graphing calculators were regularly used and, in one instance, "they helped" to trigger an intensive mathematical debate. In this paper, we will briefly discuss the relation of technology and cognition, present the data related to the debate mentioned above, and link this data to the theoretical ideas presented.

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TECHNOLOGY AND COGNITION

In this article, we will base our discussion about the relationship between technology and cognition on the work of Tikhomirov (1981). This author describes three theories which relate computers and human activity: substitution, supplementation and reorganization. In the theory of substitution, as suggested by the name, computers are seen as substituting humans since they have the capability to solve problems which previously were solvable only by humans. Tikhomirov rejects this theory, arguing that the heuristic mechanisms used by computers and humans to solve problems are significantly different.

The supplementation theory is described by Tikhomirov (1981) as seeing the computer as a complement to humans, increasing the capability and speed of human beings to perform given tasks. Tikhomirov (1981) cites information process theory as the basis of this argument, which is based on the idea that "complex processes of thought consist of elementary processes of symbol manipulation" (pp. 260). Tikhomirov (1981) criticizes this theory which views thinking as the activity of solving problems, arguing that thinking involves not only solving problems but also formulating them:

> "the formulation and the attainment of goals are among the most important manifestations of thinking activity. On the other hand, the conditions in which a goal is formulated are not always 'defined' ... Consequently, thinking is not the simple solution of problems: it also involves formulating them" (Tikhomirov (1981), pp. 261)

Tikhomirov (1981) goes deeper in his criticism of information process theory, suggesting that it does not take into account meanings that are given to the manipulated symbols and other important characteristics of solving and formulating a problem such as human values. He proposes that computers should be seen as reorganizing human activity and that emphasis should be given to human-computer systems and to problems which can be solved by them. Tikhomirov (1981) proposes, in Vygotskyan fashion, that a tool is not just added to a human being but actually reorganizes human activity: "as a result of using computers, a transformation of human activity occurs, and new forms of activity emerge" (pp. 271). In Tikhomirov's view, computers play a role similar to language in Vygotskyan theory, representing a different way of regulating human intellectual activity. Human-computer systems lead to new forms of teacher-student relationships and can suggest new ways of legitimating and justifying findings in the classroom (Tikhomirov, 1981; Borba, 1994).

We believe that tasks designed for educational practices have to take into account discussions such as the one outlined above. For instance, if one sees the computer as just a supplement, one may be inclined to design tasks which are similar to those designed, to be solved without computers, restricting the use of computers (or portable computers such as the graphing calculators) to verification of results or illustration of a given topic. In our research program, we have attempted to use tasks which take advantage of these new resources. Elsewhere (Borba, 1997) we have discussed classrooms in which the graphing calculator has a central role in students' discussions and the reorganization of their thinking. In this paper, we will present an example that suggests that even when the calculator is not being used by all the students (the students are not using it at every moment), the graphing calculator, the tasks and the environment generated has led to such a reorganization of thinking.

THE RESEARCH

This study is part of a research program of our research group (GPIMEM) at the State University of São Paulo at Rio Claro (UNESP-Rio Claro) which investigates the relationship between media and mathematics education at different school levels. This particular study has been going on for the last five years with first-year biology majors who take their only mathematics course in their first semester at the university. This course meets four hours per week (30 to 35 sessions of two hours each). There are about 44 students enrolled in the course each year. The first author of this paper was the professor of the course during all five years while the second author was a research assistant who sat in the course at all the sessions during the last three years.

Although it is a college-level course, one should think of it as an advanced high school course in which functions are analyzed more in-depth, derivative is introduced and notions of integrals are sketched. Thirty percent of the students' grade is based on what we call the *modeling approach*. In such an approach, students work in groups studying a topic of their choice. The professor collaborates with all the groups in an approach which is similar to the one described by Vithal, Christiansen & Skovsmose (1995). They were also told that it would be desirable, but not necessary, for their investigation to somehow involve the main theme of the course, functions, which all of them had been introduced to in high school. Students were expected to present partial and final written versions of their work and to make oral presentations to the class. Examples from this part of the class can be found in Borba (1997).

Parallel to this type of work, we have developed the *calculator-experimental approach*, in which the graphing calculator was used as a vehicle for experimentation by students in response to assigned tasks. Twenty Casio fx-8700 were available during each class. The classes included debates about students' findings as well as lectures by the professor and textbook work. Students were expected to write reports on their group activities involving the calculator. The data reported in this study came from this part of the course.

The data collected included written examinations, written group reports about their modeling work, and written reports about their group work with the calculators using the experimental approach. Beginning in the third year, the parts of the class involving group work with the calculators and group presentations about students' modeling projects were videotaped. Reflexive notes were taken by the professor at the end of each class and by the second author of the paper. Data was analyzed by us and by the Research Group on Technology and Mathematics Education at the Graduate Program of Mathematics Education at UNESP-Rio Claro, after several parts of the videotape were transcribed.

RESULTS

As part of the experimental approach we developed activities in which students used the calculator to think about the distance between two points, about the equation of the straight line, and about the relationship between coefficients and graphs of different families of functions such as: linear, quadratic, logarithmic and exponential. We also stressed the notion of function as a way of modeling data, when students analyzed a variation of a problem suggested by Schaufele & Zumoff (1995) in which coal consumption is modeled by functions.

After this part, we have used, during the last three years, the following task as a means of introducing the notion of derivative: is it possible to make the graph of $y=x^2$ using just straight lines? This task was the start of the data we will present regarding the reorganization of thinking and the intensification of the mathematical discussion in the classroom. As this task was given for students they were divided into small groups - two to four participants - and a lively discussion began that lasted about ten minutes. Following this, discussion among the entire class started led by some of the participants. As this discussion was occurring, many participants continued to work with their graphing calculators. The first major input was made by Iris (*Ir*) and her group who explained their approach to the problem; it consisted of considering the positive side of $y=x^2$, taking the points (0,0), (2,4) and (3,9) and then, by trial and error, finding the straight lines which went through those points. They also claimed that they could draw parabolas - made out of straight lines - more precisely if they had taken more points. Camila (*C*) then entered into the discussion and added that she had taken the points (2,4) and (3,9) and had calculated $\Delta y/\Delta x$ to find 'a' in $y=ax+b$; she used algebraic calculation with paper and pencil to calculate the value of 'b'. Camila and her group had not used the calculator to find the equation $y=5x-6$, except as a way of checking whether their algebra development looked the way they thought it should in the graphing calculator. At this time, the following dialogue took place:

Fernanda (F): professor, I don't think we will get a parabola just right, because if it is a curve ... it will always be pieced together, there will be a bump
Professor (P): there will be a bump, hum ...?
F: it will not be complete, maybe in the calculator looks like [it is smooth] . . . no, I don't think so, if you stop and think, even if there were very short straight lines, there will be the little pieces to be put together ... because it is straight line, do you understand? And the thing [the parabola] is curve, and I don't think it will be possible

It seems that *F* is uncomfortable with the method presented by both *Ir* and *C* since they see that if we keep taking smaller Δx, we will eventually be able to answer the question posed by the professor, while *F* thinks that this method will never lead to a satisfactory answer since there will always be some bumps left. The discussion continued:

Ir: But professor, what about if I put a straight line at the little bump.

F: But then you are going to smooth out the bumps, but there will be points, we are going to have only points on top of the curve.
I: But if you smooth them out out with straight lines, you won't have bumps
F: But then you will end up having dots, and this is not a straight line, then there are points to fill out ... the parabola, I think. . . .
Mayra (Ma): what I thought was the following , so . ., on each point of the parabola, we could find a straight line which passes by this point.
F: But then we are going to have points, not straight lines.

The above debate is interrupted by about ten different people speaking at the same time; it is impossible to understand what many students are trying to say except for the attempt of the professor to get one student to speak at a time. The professor was eventually successful in his attempt. But before we continue to present more data, we would like to emphasize that in the above excerpt, *F* reacts to *Ir's* solution since she believes having points instead of lines is not a fair solution for the proposed problem. *Ma* tries to solve the problem by suggesting they find a straight line which passes by this point. After the disruption of the debate, *Ma* explains her idea better, bringing new issues to the discussion:

Ma: for example, if we get 30,000 points in the parabola, . . 30,000 straight lines which pass by the parabola . .
F: then there are many straight lines, but to fill out . . the bumps, it is going to get to another point in which there will be only one point . . .
Ir: but he [the professor] said straight lines, he didn't say whole straight lines
F: but what defines a straight line is not only a point

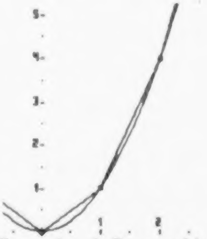

Figure 1 - A figure which resembles the one F drew in her written report.

There is a new "eruption" in the class as everybody speaks at the same time. It seems that *F* is working with line segments, or straight lines with a restricted interval (see figure 1) while *Ir* is thinking with straight lines with the domain D=R. The interpretation of F's work is also corroborated by a drawing she made in her written report about her activities during the day. By working with line segments, *F* seems to find that at the end she will not have a line, but a point, while *Ir* thinks that she still has a line if it is tangent to the parabola. *F* seems to also stick to reasoning developed in previous tasks in which the notion that we need to have two points to generate a straight line was stressed. She wants, at this point, to have these two points on the parabola. It is important to know that this is the interpretation of the researchers looking back at the experiment, since the professor, at the time of the class, had no awareness of what we believe now was the root of the problem. *F* thinks that the other solution is some sort of cheating, since the task presented mentions lines and not single points, as the next interaction shows:

F: ...in order to have a straight line I need many points, right? and I believe that to fill out a parabola we would need a straight line with many points inside the parabola....I don't understand the idea of having a straight line . . . with just one point belonging to the parabola, do you understand?

Tonini (T): professor, wouldn't it be the case that the points which go outside ...the parabola are so insignificant in relation to the scale of the graph ... that they would be negligible ... could they be negligible? ... If you magnify the scale of the graph and did a giant parabola [makes a parabola-like shape with his arms] so you could see 'the milimeters' of all those points, of . . . those bumps of the straight line, wouldn't it be so insignificant and therefore negligible...?

The professor was trying to let the discussion flow and his comments thus had a management tone, bringing the class back to organized discussion twice. He had also indicated that he was happy with the nature of the discussion about the task presented and the issues related to continuity presented by *Ma*. He attempted to summarize the discussion up to that point, without great success, but he was able to make *Ma* be more explicit about her ideas:

Ma: then, this is what I thought: for each point of the parabola, you can have a tangent line . . .

F repeats her idea about points being different from lines, and T gets back into the debate:

T: what she [Ma] talked about, the 30,000 points that the straight line passes through ... that a point of the straight line can pass by the parabola, correct? ... and if a little bump shows up...it can be neglected...

The professor was interested in exploring the idea of 30,000 points as a way of passing from secant lines to tangent lines and introducing derivative, even though he was aware that *Ma*'s notion resembled the idea of a continuous curve being similar to a necklace as described by Goldenberg & Kliman (1990). *Ma* introduces for the first time, in an explicit manner, the notion of tangent line, and another student characterizes it as touching the parabola just in one point but "staying outside the curve". Many students bring in arguments which have been presented before, and the discussion starts to go in circles. One exception is an idea brought by one student which talks more about zoom and microscope, extending the metaphor from the graphing calculator to a biological tool. The professor notes that class time is about ending and tries to summarize the issues raised by the students, using the overhead projector and a calculator to give his interpretation of what had happened. He also asks the students to bring to the next class answers to the question, Is there a more precise method of drawing a parabola with straight lines?. He had in mind getting back to the discussion about tangent lines. He leads an argument that if *Ir* and *Ma*'s ideas are used, they can get to a tangent line, and raises the question of how to calculate the slope of this straight line given just one point instead of two as they had in some of the early classes. In the next class he introduces derivative, without using the terminology yet, using the informal idea of limits and discussing the possibility of making Δx so small ($\Delta x_{->0}$) in $\Delta y/\Delta x$. In the next section, we will discuss how the data presented is connected to the ideas presented before.

DISCUSSION

The discussion led by *Ma*, *F*, *Ir*, *C* and T illustrates how rich discussion about conceptual issues can take place in a mathematics classroom where an experimental approach is used. We want to claim that, although the graphing calculator was not being used all the time by the students, we can say that human-graphing-calculator systems were in action during that class. For instance, without the graphing calculator, *Ir* could have not used her trial and error approach - and then tried to connect them to other approaches developed by her colleagues - to find straight lines which connect two points belonging to the parabola.

We also want to claim that the use of this medium did not suppress the use of other media in the classroom: orality and "paper and pencil" were some of the media used to structure the discussion which took place. In other words, the reorganization of thinking proposed by Tikhomirov is just an abbreviation of what is taking place, since as *F* has shown, she thought with the calculator when she came up with the idea of the impossibility of drawing a parabola with straight lines, but she also used paper and pencil and orality in order to structure her arguments. Other students, *C* and her group, used the graphing calculator "just to check their result" in a way which is similar to the supplementation theory. In other words, humans using a "new medium" - according to our interpretation of reorganization theory - can also use it, occasionally, in a way which resembles the supplementation theory.

Finally there is another dimension of reorganization of cognition which was not dealt with by Tikhomirov, probably because computers were very different when he developed his theoretical discussion. We want to talk about the metaphors used by students, such as T, during the debate which were derived from his previous use of the graphing calculator. He talks about scale, about bumps being negligible in a way which suggests that he had incorporated his previous use of the graphing calculator in other tasks, and that we can think of a human-computer system being the actor of his argumentation as well. We can also think that the student who connected the zoom of the calculator with a microscope was thinking with these instruments as well even though the microscope was not present there. In this case, he implicitly, showed the idea of local straightness of a differentiable curve, as suggested by Tall (1991), which is strongly linked to the media.

In this paper, we presented a way of understanding the relationship between computers and cognition, some original data which was collected within the activities of our research group (GPIMEM). We summarized the ideas of Tikhomirov according to our interpretation and confronted them with the data presented. In doing so, we raised some issues about the limitations and convenience of his ideas regarding the links between computer/graphing calculator and cognition. However, we do agree with his main idea regarding reorganization as we discussed above, and we believe that the data presented could not corroborate supplementation theory since the role of the calculator was, for the most part, not peripheral.

Finally, we would like to add that the data presented is a relevant example which could also be analyzed with respect to other issues, such as: the nature of the interaction between professor and the students using some of the ideas about voice and perspective (Confrey, 1993); the relationship between the knowledge being constructed about tangent lines in the classroom and the way academic mathematics organized such notions throughout history; and Levy's (1993) notion of collective thinking in which he proposes that cognition is a collective entity not only formed by humans and computers but also by other non-human actors. We believe that developing these ideas in other papers will help us to build a broader picture of the nature of the mathematics which emerge from experimental approaches within mathematics education.

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PRE-ALGEBRA: A COGNITIVE PERSPECTIVE[1]

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Research in learning algebra has demonstrated the link between arithmetic and algebra, identified a gap in this transition, and proposed a pre-algebra level. This paper reports on a longitudinal study discussing this transition from a cognitive perspective. Thirty-three students in grades 7, 8, and 9 participated. Students' readiness for algebra instruction and linear equations in terms of prerequisite knowledge was explored in order to determine what constituted a pre-algebraic level of understanding. A two-path model for the transition from arithmetic to pre-algebra to algebra is proposed and students' understanding of relevant knowledge is discussed. Results showed a developmental sequence that appears to fit the model.

Pre-algebraic and Algebraic Understanding

Some research in early algebra teaching and learning has focused on the transition from arithmetic to algebra and the difficulties in developing algebraic concepts caused by a cognitive gap (Herscovics & Linchevski, 1994) or didactic cut (Filloy & Rojano, 1989). It is suggested that the cognitive gap/didactic cut is located between the knowledge required to solve arithmetic equations, by inverting or undoing, and the knowledge required to solve algebraic equations by operating on or with the unknown or variable. Linchevski and Herscovics (1996) proposed that students could not operate spontaneously on or with the unknown and that grouping algebraic terms is not a simple problem. They also argued that students viewed algebraic expressions intuitively as computational processes (cf. Sfard & Linchevski, 1994) and suggested that in teaching, instead of moving from variable to expression to equation, arithmetical solution of linear equations might be more suitable initially for learning to operate on or with the unknown. Filloy and Rojano (1989) believe such concerns point to the need for an operational level of 'pre-algebraic knowledge' between arithmetic and algebra.

However, what is not clear in the literature, is the level of understanding denoted by pre-algebra. Herscovics and Linchevski (1994) consider that the cognitive gap between arithmetic and algebra defines a level of pre-algebra and regard this as "involving those intuitive algebraic ideas stemming from the presence of an unknown in a first degree equation" (p. 75). Linchevski (1995) provided an explanation for pre-algebra as incorporating substitution of numbers for letters; dealing with equivalent equations through substitution; and allowing students to build cognitive schemes through reflective activity and spontaneous procedures.

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Bell (1996) proposed six hypotheses about algebraic thought. These included: resolution of complex arithmetic problems by step-by-step methods working from given data to unknowns or by global perceptions and use of multiple arithmetic relations; coding and using systematic general methods for different problems; recognition and use of general properties of the number system and its operations; and use of a manipulable symbolic language to aid this work. We believe that these hypotheses are concerned both with pre-algebraic and algebraic thought.

It is our contention that the transition from arithmetic to algebra involves a move from functioning arithmetically with numbers and operations, to operating at a pre-algebraic level which involves intuitive algebraic ideas and solution of linear equations with one unknown using inverse procedures, to operating algebraically which encompasses a series of operations and more than one variable and the understanding of relationships expressed in general and simplified form.

The Role of Unknown

As stated previously, we consider that understanding the concept of the unknown and solving to find the unknown in an equation, constitutes in part a pre-algebraic level of understanding. Panizza, Sadovsky, and Sessa (1997) suggest that the notion of unknown may become an epistemological obstacle when trying to conceptualise the notion of variable. However others, such as Rojano and Sutherland (1997), believe that working with the unknown, both symbolically and numerically, will allow students to accept the idea of operating with an unknown quantity. Graham and Thomas (1997) maintain that allowing students to gain an appreciation of letters as labelled stores will help develop an understanding that will improve assimilation of later concepts.

A study conducted by Ursini and Trigueros (1997) found that college students had difficulty discriminating between variable as unknown and variable as generalised number and propose that understanding of variable as unknown implies: recognising and identifying in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem; recognising the symbol that appears in an equation as an object that represents specific values that can be determined by considering the given restrictions; the ability to substitute for the variable, the value or values that make the equation true; determining the unknown quantity that appears in equations or problems by performing the required arithmetic and/or algebraic operations.

Analysis of Sequence for Learning Linear Equations

Complex linear equations in algebra such as $2x+3=11$ include three crucial components: an equals sign, a series of more than one operation, and a variable 'x'. We are describing these equations as complex, because they include more than one operation, as opposed to binary operations such as $x+5=6$. We propose a two path model for learning complex algebra where binary arithmetic ($2+3=5$) operations, complex arithmetic ($35\div7+8=13$) and complex pre-algebraic operations [$3(x+7)=24$] are necessary components of one path and binary arithmetic operations ($2+3=5$), binary pre-algebraic ($x+7=16$) and binary algebraic ($x+y=12$) operations are necessary components of a second path. This means that understanding binary operations, such as $2x$ and $x \pm 3$, should be a prerequisite to understanding $2x+3=11$ as should application of operational laws to series of operations. Additionally, we suggest that equations such as $x+7=16$ require solution procedures of a pre-algebraic nature which, at the lowest level, comprise use of inverse arithmetical procedures to find the unknown. The two path model also assumes that learning linear algebraic equations will be facilitated by understanding isomorphic structures in complex arithmetic.

Evidence from the developmental literature supports this model in that it suggests acquisition of pre-algebraic and algebraic concepts in the following order: one occurrence of the unknown in binary operations, a series of operations on and with numbers and the unknown, multiples of the unknown, acceptance of lack of closure and immediate solution with a series of operations on the unknown, and finally relationships between two variables and operations on them.

The purpose of our study was to explore students' early understandings of algebraic concepts as they moved from arithmetic to algebraic. This was to determine: (a) the validity of the two-path model of sequential development of algebraic understanding and (b) what constituted a pre-algebraic level of understanding. Results from pilot work were published in Boulton-Lewis et al. (1995), Boulton-Lewis, et al., (in press), and for the pilot work and the first year of the study in Boulton-Lewis et al. (1997) and Cooper et al. (1997). This paper presents results for the three-years of the longitudinal study.

Method

Sample

The sample comprised 33 students who were tested in the first year in four state primary schools in Brisbane. These were feeder schools for the high school where, in the second and third years of the study, the students were in grade 8 and then grade 9. Generally these schools were in a middle socio-economic area.

Interviews were conducted with the grade 7 students before any formal algebra instruction took place and with grade 8 students after they had received instruction in operational laws, use of brackets, and solution of arithmetic word and number problems. Grade 9 students were interviewed after they had learned about an 'unknown' in a linear equation and solution of a linear equation using balance procedures.

Materials and Tasks

Students were presented with expressions and equations written on cards and asked questions that investigated: commutative and inverse laws; order of operations [$(32+(12\text{x}8)\div3$]; meaning of equals in an unfinished equation ($28\div7+20=$) and in a complete equation ($28\div7+20=60-36$); meaning of unknown ($\square+5=9$; $x+7=16$; $3x=12$) and variable ($\square+5$; $3x$);); and solution of linear equations [$3x+7=22$; $3(x+7)=24$].

Procedure

Students were interviewed individually and videotaped. They were encouraged to complete each task however if they could not respond the interviewer proceeded to the next task.

Analysis

Interviews were transcribed and analysed to identify key categories. The NUD*IST program (Richards & Richards, 1994) was used to classify response protocols under these categories and further sub-categories. Responses for laws and order of operations were categorised as *satisfactory* or *unsatisfactory* as a basis for learning algebra. Responses for the other tasks were categorised as *inappropriate*, if they indicated a lack of knowledge required for the task; as *arithmetic*, if they focussed on arithmetical procedures and numerical answers); as *pre-algebraic*, if they evidenced understanding between arithmetic procedures and intuitive algebraic ideas and used inverse procedures; and as *algebraic*, if they evidenced recognition of relationships expressed in simplified form and recognition and use of general properties of the number system and its operations.

Results

The summary results below will be illustrated by Tables and examples of responses at the conference presentation.

Commutative and Inverse Laws and Order of Operations

In grades 7 and 8 the majority of students (19 and 17 respectively) could not explain *commutativity* of addition and multiplication satisfactorily. However by grade 9, 25 students gave a satisfactory explanation for commutativity. *Inverse operations* were explained satisfactorily by the majority of students in each grade (26, 30, and 33 respectively) and by grades 8 and 9 most students explained *order of operations* satisfactorily (26 and 23 respectively compared with only nine satisfactory explanations in grade 7.

Meaning of Equals

In each grade, the majority of students explained '=' in $28 \div 7 + 20 =$ as find the answer. Only one response in grade 8 and three responses in grade 9 evidenced knowledge that '=' denoted an equivalence relationship when they stated that both sides had to be equal. For '=' in $28 \div 7 + 20 = 60 - 36$, the majority of responses moved from arithmetic in grade 7 when students (19) stated equals meant the answer, to arithmetic (12) or algebraic (12) in grade 8 as students explained equals as either the answer or denoting equivalence, to algebraic in grade 9 with most students (19) explaining equals as equivalence or showing a balanced equation.

Meaning of Unknown and Variable

The majority of students in each grade indicated that □, in $\square + 5 = 9$ (16, 22, and 21 respectively), and x in $x + 7 = 16$ (18, 24, and 26 respectively) represented an unknown number. However when x was presented in $3x = 12$ in grade 7 most students (18) did not know what this meant and gave an inappropriate explanation. In grade 8 most students explained concatenated x either arithmetically as a times sign (12) or pre-algebraically as an unknown number (12). In grade 9 most students' (25) explanations for x in $3x = 12$ were pre-algebraic stating that x was an unknown number.

For meaning of variable in grade 7 most students (18) stated pre-algebraically that □ in □ + 5 represented an unknown number with another eight students stating algebraically it was any number. Five students gave inappropriate responses and two stated it was the answer. In grade 8 and grade 9 the majority of students stated pre-algebraically that □ was an unknown number (15 and 18 respectively) or algebraically that is was any number (14 each grade). In grade 9 there was only one arithmetic response and this indicated □ was the answer. For x in $3x$ most students in grades 7 (19) and 8 (15) responded arithmetically that it was a times (multiplication) sign. However by grade 9 the majority of students (17) stated pre-algebraically that it was an unknown number and a further 10 students responded algebraically that it represented any number.

Solution of Linear Equations

The majority of students in grades 7 (14) and 8 (13), as one would expect, did not know how to solve $3x + 7 = 22$. Eight students in grade 7 and 10 students in grade 8 used inverse arithmetic processes to find what they believed was missing after x because they interpreted x as a 'times' sign. Nine students in grade 7 and 10 students in grade 8 used inverse processes to solve for x which was categorised as pre-algebraic. By grade 9 most students (23) solved $3x + 7 = 22$ pre-algebraically by using inverse processes. Two students did not know how to solve the equation, two used an incomplete prealgebraic balance method which entailed balancing the equation by taking 7 from both sides, however at this point the students then said that was 15 divided by 3 which is 5, rather than dividing each side by 3. Six students solved by using a complete balance procedure which was categorised as algebraic.

For $3(x + 7) = 24$ the majority of students (27) in grade 7 did not know how to solve the equation, while six students used a pre-algebraic inverse procedure. By grade 8 the majority of students still did not know how to solve the equation, however 12 students did use a pre-algebraic inverse procedure. Six students used arithmetic processes: two used the inverse by finding the space after the x and four used trial and error. By grade 9 most students (19) used pre-algebraic inverse processes to solve $3(x + 7) = 24$. Another four responses were pre-algebraic: three used an incomplete balance process and one student used the balance method incorrectly. Six responses were inappropriate, one student used a trial and error arithmetic process, and three students used a complete balance procedure which was categorised as algebraic.

Discussion

By grade 9 most students had sufficient understanding of the commutative law to apply this to linear equations, the majority of students also displayed a satisfactory understanding of inverse procedures and of the correct order of operations. Herscovics and Linchevski, (1994) include understanding the order of operations as indicative of arithmetic functioning. These results indicate that by grade 9 most students had satisfactory arithmetic understanding to enable them to apply these principles to algebra

For equals in the unfinished equation, the majority of students each year indicated an arithmetic understanding by stating it meant find the answer. However for the finished equation, most understanding of equals moved from arithmetic in grade 7, to arithmetic or algebraic in grade 8, with most students in grade 9 stating algebraically that '=' denoted an equal or balanced relationship. However in all three grades almost one third of the students interpreted '=' pre algebraically. Kieran (1981) noted that students require an equivalence understanding of equals to operate algebraically. By grade 9, 19 students demonstrated an equivalence understanding, however there were still 14 students who were operating at either a pre-algebraic or arithmetic level. This suggests that while students' knowledge of '=' had developed over the years, there was still a substantial number of students who did not understand '=' in an algebraic sense and would need to learn the concept of equivalence. Providing explicit instruction of equals at a pre-algebraic level, that is that each side is the same, may help bridge the gap between arithmetic and algebraic understanding of equals.

Most students, over the three years, knew that □ in the expression and equation represented an unknown number. In the expression, in particular, this is indicative of a pre-algebraic level of understanding as □ could be interpreted algebraically as representing 'any number'. In grade 8 and 9, 14 students did explain □ as any number and as understanding emerged in grade 9 some said that it was a variable. These results indicate that understanding □ initially as an unknown number appears to be a suitable foundation from which to introduce the concept of any number or variable. Similarly by grade 9 most students explained x (not concatenated) in the equations as an unknown number.

Understanding of x in $3x$ was a more cognitively demanding task. Most students in grades 7 and 8 did not understand concatenated x and hence could not solve the linear equations. However by grade 9, after students had been instructed in concatenation and use of inverse or balance procedures to solve an equation, most chose to use inverse pre-algebraic procedures to solve linear equations while a smaller number of students used balance procedures successfully. Sfard and Linchevski (1994) view the process of solving to find an unknown by reversing procedures or backtracking, as early algebraic thinking. We suggest that this solution process would be more appropriately placed at a pre-algebraic level of functioning.

Conclusion

Overall, results for the three years of this study, support the contention that there is a case for focussing on an explicit pre-algebraic level of understanding. This was particularly evident in students' explanations for equals in a finished equation, □ and x in the expressions, and solution of the linear equations. The findings also support the sequence of instruction, as proposed in the model, that understanding of binary operations such as $3x$ is a prerequisite to solution of complex algebraic equations. We suggest that as arithmetic procedures are applied intuitively they should constitute a sound basis for pre-algebraic instruction. Finally we propose that pre-algebra should include instruction in: operational laws; equals as equality of sides leading to equivalence; solution of binary and complex equations using inverse procedures; use of letters to represent unknowns as distinct from variables; concatenation; and should be based on students' arithmetic as well as intuitive algebraic knowledge.

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THE RIGHT BAGGAGE?

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Abstract

There has been a great deal of research into aspects of mathematics anxiety and learners negative experiences and the effects of collecting this baggage that is carried onto future experiences with the learner. Are the experiences and therefore the baggage collected by those who are successful in studying mathematics different? This paper addresses the issue of what constitutes the 'right baggage' collected from early experiences that positively influence attitudes and achievement in mathematics. Using oral history as the method for collecting those early experiences this paper describes the results from interviews of mathematicians, raising questions for mathematics educators.

Introduction

The baggage accumulated from early experiences is carried with the learner and influences later attitudes to learning mathematics. Studies of people's learning of mathematics have focused on those who had been unsuccessful, fearful or disinterested. Briggs and Crook (1991) investigated student teachers' attitudes to mathematics and found they used words like 'totally devastated,' 'frustrated', 'embarrassed', 'failing' and 'terrified' to describe their memories of learning mathematics. Yet there are those who enjoy the subject, are successful and go on study mathematics in great depth. A quotation from Russell gives a flavour of how some people have very positive feelings about mathematics; "At the age of 11, I began Euclid...this was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world," (Buxton, 1981, p.17) What are the experiences of those who are successful? Do they have different early learning experiences?

Oral History

Oral history is the collection of an oral record of interviews with individuals to investigate specific events or groups of people through a focus on their lives. The material generated provides a historical narrative. For the researcher this creates advantages and disadvantages. Derived from human perception it is subjective but being oral allows the researcher to challenge subjectivity probing beneath/beyond the subjective responses of the interviewees. This does not mean that the researcher accepts the material as objective as this approach is not chosen to identify specific truths. The subjectivity itself can be more revealing, the way people remember what happened and the effect that has is perhaps more important than what actually happened.

Oral history has been used to study a wide range of historical events...'and proved to be a training ground for imaginative interpretation, rather than an alternative to archives....' (Niethammer, 1979,p.27). In choosing to use oral history material for research purposes the intention is to focus on general pictures being conveyed of individuals lives and the key events within them. For this study using oral history provided the opportunity to focus on the lives of mathematicians and to find out more about specific events, people and aspects of mathematics that have been of key importance to them as individuals. It also offered the opportunity to look at the positive experiences of learning mathematics when the vast majority of the literature available at present dwells on the negative aspects and experiences of mathematics.

The interviews.

There were five people interviewed for this study all of who have studied mathematics to doctoral level. The information gained during the interviews focuses on the interviewee's childhood, family structure, education and experiences of learning mathematics. The interviews were semi-structured, there were no specific questions but all covered the following areas:- Introductions, date and place of birth, family-parents-background and siblings, extended family, early childhood, school entry, primary school, secondary school, influence of teachers, influence of events, and, when mathematics and why? There were a number of supplementary areas covered due to the information given by the interviewees, for example background information about particular countries and political events during the interviewee's life time. There was 'a basic shape to guide the mind' (Thompson 1978, p. 171), and the structure trialed through use of a pilot interview. All interviews were carried out in a location familiar to the interviewees as the location of the interview is important in setting the right tone and can effect the responses recorded (Thompson, 1978, p.119).

Analysis of the interviews.

When analysing any interviews the listener must be aware of the relationship between the speaker and the material imparted.

> Like myth, memory requires a radical simplification of its subject matter. All recollections are told from a stand-point in the present. In telling, they need to make sense of the past. That demands a selecting, ordering and simplifying, a construction of coherent narrative whose logic works to draw the life story towards a fable. (Samuel and Thompson, 1990,p.8)

At the same time the interviews can be considered to contain facts plus the speaker's interpretation of their lives with emotions and personal events (Giles, 1992). As a result of this when evaluating the material a check for bias must be made. This can be assessed by focusing on the internal inconsistency, though some minor in consistencies might be expected, a tendency to fabricate information generally is likely to be present throughout the interview. What is more difficult is consistency within the selection and interpretation of material across a number of interviews. This issue is discussed at length by Ochberg (1996), he describes the process as trying..'to show what an informant accomplishes by recounting his or her history in a particular fashion' (p.98). 'The point of the interpretation is not to understand a single individual but to enlarge our conception of how sense might be made-' (p.102) In interpreting a number of life histories the information is converted from one kind of account into another, from a story into a particular argument with a specific focal event or issue. As Ochberg (1996) says ..'people do not register experience passively. Instead sense is made.' (p.112.) he goes on ...'Listening to

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them from an interpretative point of view is not demeaning. It is rather, the only way we can notice both the power and the limits of our narrators' attempts to make something of their experience-and, thereby, themselves' (p.112). Thompson (1978) list of ways oral history can be put together: 1. Single life story narrative or; 2. Collection of stories (groups of lives to portray a community) or; 3. Cross-analysis: the oral evidence is treated as a quarry from which to construct an argument The analysis in this study is focused on number three. The following are the key aspects identified across the interviews.

Parental influence.

Parental influence was a strong theme of all the interviews. In some cases it was centred on a forceful parent and their vision of what their children were going to go on to do in later life. In these cases in appeared to predominantly be the mother who organised and had the clarity of purpose as can be felt in the following:

> I know for a fact that ever since I can remember my mother had decided I was going to Oxford. (Interview with 'Thomas' 12/5/97)

Roberts (1995) describes a growing minority who was ambitious for their children, encouraging them to stay on at school and going on to higher education. Interestingly Roberts highlights that it was the mother's attitude that determined the type and quality of education they received. For 'Thomas' and 'Sam' this was clearly the case as their mother's were the family organisers. This is supported by evidence collected by Roberts (1995, p.138) ...' mothers regarding the needs of their children as paramount'. as in the following:

> My mother was a very determined person, she was the one who organised us all, with my father's backing...She always made sure her job was tailored round being there for us and making sure we did what we were supposed to do. (Interview with 'Sam' 16/7/97)

In terms of the trend in working class parents' children going on to University and a correlation between parental aspirations and children's academic achievement then 'Thomas' and 'Harry' parents fit into this category.

> They were quite supportive they thought of education as a good thing because they hadn't had it. The people they admired and respected had had it, had got on further so they saw it as something valuable for its own sake but also as kind of economic security for my future.
> (Interview with 'Harry' 5/7/97)

Yet in 'Sam's' interview the feeling was still prevalent as father was described as 'coming from humble beginnings'. All those interviewed mentioned support with school work, this was sometimes overtly providing practice for tests.

> Do well academically and that was the number one thing as we were children and we were given every sort of help and encouragement like

> if there was a test coming up we were helped (Interview with 'Sam' 16/7/97)

Or it could mean providing extra classes to support the usual school work which 'Charles' mentioned on a number of occasions. Above all these parents had confidence in their children's abilities and set high expectations for their children. For all those interviewed this was positive, no one described this as being pushed or struggling and becoming concerned about pressures to succeed. For at least one of those interviewed the parental influence appeared stronger when decisions were to be made about study at University level:

> There was some possibility that I might do engineering because those were the jobs that were opening up and I quite liked it but I couldn't decide so my father said do mathematics because you can always go on do something else afterwards. ..(Interview with 'Charles' 4/7/97)

Roberts (1995, pp.51-3) details parental involvement in career and job choices as being the norm for children brought up in the inter-war and post-Second World War. There was never any question of making decisions other than the ones your parents made, it was a case of going along with their decisions. Other members of the family influenced behaviour and learning at the same time sign posting possible directions for future studying. This brings in the notion of role models and children aspiring to emulate those they held in high regard. There was also specific subject matter that could be learnt from these members of family.

> When I was 11 she (my aunt) taught me and my cousin, her son, trigonometry,....I remember managing to work out what these things were and enjoying that and she introduced me to the x when I was 11 and that certainly had an effect. I wanted to become a scientist because she was and my mother also had been...(Interview with 'Edward' 29/7/97)

Role of the teacher.

In studies that focused on the negative attitudes to learning mathematics the teacher played a significant part in people's recollections of events that formed their general view of mathematics. Briggs and Crook (1991) cite a number of experiences where people were humiliated by the teacher in front of the class and many others related a lack of sensitivity on the part of the teacher.

> The teacher thought I had cheated by looking up the answers in the back of the book, because my answers were exactly right. Although she didn't accuse me, she called everyone up and tore the answers out from our books. I never forgave her for humiliating me like that (p.49)

It is not surprising that teachers play a central role in many of the interviewees' experiences of learning mathematics and the examples that follow show the positive influence the teachers had those interviewed. For Thomas being told what he needed to do was an encouragement not a put down.

> There was a guy who had this quality about him, he loved his mathematics. He gave no praise to anybody, he never told me how good I was, he always told me how bad I was, he told me what more I needed to do to get there. (Interview with 'Thomas' 12/5/97)

A new teacher to a school offered a different role model, brought different ideas with them and opened up possibilities that were previously not available.

> He produced the first maths sixth form ......Obviously there was something dormant there, but I think it could equally have been another subject maybe I had some special affection for mathematics. (Interview with 'Harry' 5/7/97)

Different ways of teaching become a significant issue and remembered when considering ones own teaching methods.

> We had a teacher who taught us geometry and he used to use more modern techniques like ... set us problems and ask us to talk to our neighbours about it this, he used to set us problems where he hadn't previously shown us the solution. (Interview with 'Edward' 29/7/97)

One of the things about significant people in some one's life is that their influence pervades the whole of their lives at the time rather than just influencing a small part as the following shows.

> 17th period we did non mathematics, so over two years we listened to music, studied art and talked about current affairs, we talked about philosophy, all sorts of things I didn't have in my home background....He opened up a new world to me in that formative adolescent period...a genius of a teacher had an enormous effect on my life and my self esteem (Interview with 'Harry' 5/7/97)

Not all experiences have to be positive to influence and motivate the learner.

> When I was about 10, I wanted to learn algebra. There was a girl in my class in my primary school and she was allowed to read this more advanced book and I told the teacher I wanted to be able to study this more advanced book and the teacher wouldn't let me. One day at lunch time I went to the cupboard and I stole this book out of the cupboard and the next day the teacher hauled me up in front of the class and demanded in a loud voice where this book was. This little girl who was allowed to read this advanced book happened to be away that day so I said she told me I could take the book from the cupboard (Interview with 'Edward' 29/7/97)

The mathematics is the thing.

For many people who have struggled with mathematics it can be the nature of mathematics or specific aspects of the subject that cause significant difficulties. Bell et al. (1983) point towards key areas of mathematics that cause difficulty. For some of those interviewed in this project the introduction to specific aspects of mathematics provided the key to shifting the focus of their attention towards mathematics or showed an early indication of where their interests might lie in the future. Or mathematics appeared to get easier and so became the selected subject for study and gave some insights into the mathematicians view of mathematics as opposed to views we might hold of the subject!

> I loved doing it, I loved the shape, I loved the power of it, I loved the way that it didn't matter which way you did it there was one truth inside. What you were trying was to seek this distilled essence that was in there that was so pure and beautiful. I got exquisite pleasure out of mathematics. (Interview with 'Thomas' 12/5/97)

> There's something about patterns that sort of attracts me but then it's also about finding the links between them so may be you have a numerical pattern and some algebraic thing, combinatorial thing kind of tracing and this thing tells you about this one...(Interview with 'Sam' 16/7/97)

> (Mathematical proof)...that is what started my interest in mathematics and I am sad that they don't they don't do this sort of thing any more. It is very enjoyable, perhaps not by everybody. I am an algebraist and this is what interests me, not only to we have the rule but we know why it works. .(Interview with 'Charles' 4/7/97)

> It's a feeling that kind of these abstract things that all seem quite mysterious but by thinking about them you can make sense of it so it's a sense of the ability to dominate over the mysterious by giving a lot of thought (Interview with 'Edward' 29/7/97)

As with anyone whom has a passion for a subject these people see many interesting and exciting things in mathematics that challenges their skills and maintaining their continued enthusiasm for the subject.

Turning points.

At the University level of study there were clear turning points for two of the interviewees. For the first time spent boycotting studies resulted in a decision that shaped the events to come.

> I realised that the only way I could possibly catch-up was by doing mathematics, not by doing these four applied subjects at Cambridge, so I became a mathematician. (Interview with 'Edward' 29/7/97)

For the other a change to an engineering career didn't happen as a result of moving to England.

> Part of the reason why I didn't become an engineer was I came to this country and mathematics had a much higher status here than engineering (Interview with 'Charles' 4/7/97).

Motivation.

What actually motivates people to do particular things or to decide a specific course of action is probably individually based yet it is possible to see some trends in the areas spoken about in the interviews. As Cyril Willis recounts in Humphries (1981, p.58) at least two of those interviewed felt they had to succeed for their parents.

> My mother was always there as this dominant force behind me. I was going to conquer the earth so when I was working I was always aware that I had to keep going to conquer the earth...(Interview with 'Thomas' 12/5/97)

The challenge that mathematics provided was a strong motivation for most of those interviewed coupled with the fact that there were answers to found that could be evaluated.

> I think a lot of the reason I worked hard was wanting approval and not wanting to be criticised to the point where anybody told me anything wasn't perfect it really upset me so I would work to see that it was really right (Interview with 'Sam' 16/7/97)

During the early stages understanding took on less significance than getting things right.

> I learnt the routines and liked getting the right answer and I liked getting back my homework with lots of ticks on (Interview with 'Harry' 5/7/97)

Or the challenge of finishing the work:

> I was doing calculus questions, it was such a joy such excitement so that first term coming up to Christmas day I'd nearly finished. When I woke up on Christmas, I opened my presents then I put them away. I was still doing calculus and finished that book at three o'clock on Christmas day. (Interview with 'Thomas' 12/5/97)

Even at an early age for some the challenge was even more exhilarating because the journey would continue and not cease.

> You see the peak in the distance and you think I'm going to get there, you get there and then all of a sudden as you get closer you begin to realise...and you see through the mist another peak....It wasn't a

> deadening influence it was a liberating influence that I'd never sort of complete the task. (Interview with 'Thomas' 12/5/97)

Conclusions and implications?

'Oral History is not necessarily an instrument of change; it depends upon the spirit in which it is used'. (Thompson, 1978, p.2). In drawing tentative conclusions from the interviews it is possible to highlight four factors influencing the right baggage. Firstly the support of parents in the education process that is perhaps the least surprising of the findings since it is generally accepted that children do better at school if parents take an interest. Though the role of the mother in particular in the process was an aspect of this support I shall be more aware of as a result of this work. ...'The educational historian becomes concerned with the experiences of children and students as well as the problems of teachers and administrators'... (Thompson, 1978, p.6). Secondly and again unsurprisingly the role of the teacher was for most of the speakers a factor in their success. If a teacher can influence negatively then the converse should also be true. Thirdly the element of challenge at the right time was linked to the role of the teacher though not exclusively. This has implications for the teaching of mathematics if children are to be encouraged through challenges. This in itself is a challenging task as the teacher must identify the children ready for the challenge and provide just the right amount of challenge at the appropriate time!! Fourth that certain aspects of mathematics appeared to trigger a real enthusiasm for the subject setting it a part from others with which child might have been equally successful until they channelled their energies towards mathematics.

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"LEARNER-CENTRED" TEACHING AND POSSIBILITIES FOR LEARNING IN SOUTH AFRICAN MATHEMATICS CLASSROOMS

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This paper looks at the notion of learner-centred pedagogy, particularly, but not only, in the South African context. A conceptual elaboration of important aspects of learner-centredness is brought together with an analysis of two mathematics lessons in the senior secondary school. It is argued that focussing on the learner, at the expense of the teacher and the mathematics, enables teaching that may look learner-centred, but which may not enable mathematics learning. On the other hand, a focus on the mathematical problem-solving process may enable more learner-centred teaching.

Introduction

"Learner-centred" teaching is on the agenda in South Africa. It is a key principle informing curriculum development (National Department of Education, 1996), and is being presented to teachers as a cornerstone of the new "Curriculum 2005". The government's notion of "learner-centred" is rooted in constructivist theories of knowledge and in the need to recognise and affirm diversity in schools and in the society, as can be seen in the following quote: "The ways in which different cultural values and lifestyles affect the construction of knowledge should also be acknowledged and incorporated in the development and implementation of learning programmes" (National Department of Education, 1996:11).

Teacher education around the new curriculum has emphasised learner activity, participation and groupwork as central aspects of learner-centred classrooms. Teachers are encouraged to "facilitate" learning rather than provide instruction. A "paradigm shift" from past practices is urged, with the past being characterised as "teacher-centred" and encouraging of passive learners who engage in individualised, rote learning rather than creative and flexible thinking (National Department of Education, 1997:6-7). I have argued elsewhere (Brodie, 1997, 1998) that such a characterisation is untrue and unhelpful for teachers who want to improve their teaching.

Learner-centred pedagogy is not a new concept in many countries, nor in mathematics education. However its insertion into curriculum discussions in South Africa at this point enables a (re)consideration of some key issues. In particular, how might a notion of learner-centred teaching enable teachers to work for improved learning of mathematics? In trying to answer this question, I will first elaborate some important dimensions of the concept of learner-centredness. Thereafter, I will analyse two senior-secondary mathematics lessons, one where the teacher appears to be more learner-centred and another where the teacher appears less so. To do this, I will draw on Edwards' and Mercer's (1987) analyses of classroom interaction. In this paper I do not intend to present definitive answers to the question of what learner-centred mathematics teaching might look like. Rather, in drawing together some conceptual elaboration with an analysis of classroom practice, I hope to raise questions and enable discussion which will be useful for research and teaching.

"Learner-centredness"

The new, outcomes-based curriculum, "Curriculum 2005" forms part of the South African government's strategy to transform South Africa's unequal, poorly resourced and divisive education system. The new curriculum is part of the promise of "a better life for all"and is intended to develop a "prosperous, truly united, democratic and internationally competitive country with literate, creative and critical citizens, leading productive, self-fulfilled lives in a country free of violence, discrimination and prejudice"(National Department of Education, 1996:5)[1]. "Learner-centredness", as a key principle informing the new curriculum, is closely connected to outcomes-based education. Outcomes focus on the learner and express levels of competence in relation to skills and knowledge which the learner should attain. This is a notion of learner-centredness which focuses on the products of learning, the outcomes.

Juxtaposed with this product-oriented understanding of learner-centredness is the more familiar Piagetian "process" notion where children are seen as agents of their own learning, who can and will discover or construct important principles if provided with the appropriate experiences at the right time. Direct teaching can be counter-productive for children's learning, hence the terms "educator" and "facilitator" which are currently preferable to "teacher" in the South African discourse. The process-product relationship has not yet been articulated, and this may result in superficial rather than deeper manifestations of learner-centred teaching.

Using an analysis of classroom talk in British primary classrooms, Edwards and Mercer (1987) argue that teachers who attempt to allow pupil "discovery" (process) come up against the need for pupils to "discover" particular principles (products). When the pupils don't make the required discovery, teachers are forced into particular forms of discourse to help them achieve this. These include cueing pupils' responses, often quite heavily, ignoring wrong answers, and reformulating correct answers into the ongoing classroom narrative. So a focus on the pupil, at the expense of the knowledge to be learned, in the context of school learning, creates a contradiction for teachers: children should discover on their own, but they should also discover particular knowledge in particular ways.

Other critiques of a Piagetian notion of child-centredness include those of Donaldson (1978), Bruner (1996) and Walkerdine (1984). Donaldson showed that the interactional context influences children's responses to Piagetian tasks. Using Vygotsky, Bruner argues for the social and cultural situatedness of knowledge and the learner. Walkerdine argues that the notion of the child in child-centred pedagogy is a social construction, made possible by historically specific conditions in academic discourse and educational practices at the time. Walkerdine's critique in particular suggests a reconsideration of learner-centredness in the specific educational conditions in post-apartheid South Africa.

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[1] An important critique of the new curriculum is its naive view of the relationships between education and development (see Jansen, 1997). This debate cannot be entered into here.

These critiques all suggest that we cannot focus too narrowly on the learner. Although "learner-centred" is often set up in opposition to "teacher-centred", or "subject-centred", in fact the learner cannot be thought about in isolation, and the interrelation between learner, teacher and mathematics in context, is crucial in developing a useful notion of learner-centredness.

In the South African government's explication of learner-centredness we see phrases such as: "Curriculum development should ... put learners first, recognising and building on their knowledge and experience and responding to their needs" and "Motivating learners by providing them with positive learning experiences, by affirming their worth and demonstrating respect for their various languages, cultures and personal circumstances is a pre-requisite for all forms of learning and development" (National Department of Education, 1996:11). While social context and diversity are acknowledged, this understanding of learner-centredness emphasises the learner, possibly at the expense of the teacher and the mathematics. Obvious questions which can be asked of the above include: What happens when different learners' needs are in conflict, which is likely in situations of diversity? How does a teacher put all learners "first"? If the learners are "put first", what comes "second"[2]? How are the learners' present needs balanced against future needs, particularly the very important need for access to mathematical knowledge.

In my work with in-service teachers, it is my experience that they are primarily concerned with improving their pupils' learning and that they do think about what is best for their learners, in relation to the mathematics that they need to teach. Current practices, including those associated with "rote-learning" (for example chorusing and chanting, see Setati, 1998) are seen to be appropriate ways to achieve better mathematics learning, particularly in the context of large classes of underprepared pupils, learning in a language in which they (and often their teachers) are not confident. So urging teachers to "put learners first" without substantial discussion about what is meant, may become counter-productive, firstly because practices which are problematic can be seen to fit with the new rhetoric, and secondly, because in attempting to make shifts, teachers may lose the strengths of current practice.

Another, seldom discussed, aspect of learner-centredness, is that children's ideas make sense when seen in light of the pupils' own logic and perspectives. Therefore a more useful notion of learner-centred teaching is for the teacher to actively engage with learners' ideas on their own terms and in relation to the mathematics that is to be learned. Real engagement involves extending and constructively challenging pupils' ideas. Teacher mediation of the mathematics is crucial. Mediation is a complex practice, and dichotomies such as "facilitation", and "non-intervention", as opposed to "direct teaching", are not useful in delineating helpful from unhelpful practices, nor in indicating where particular teaching approaches can be improved to enable better mathematics learning. Teachers' decisions about how to enable and follow up on learner contributions are important, as are their ways of offering alternate ideas and frameworks which enable access to the discourse of mathematics.

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[2] Thanks to David Pimm for a discussion which provoked some of these questions.

In light of this discussion of aspects of learner-centredness, I will look in more detail at two mathematics lessons, one of which might be considered learner-centred in terms of pupil participation and "discovery" and a second which would probably not be. I will look at how each teacher enables and works with pupil contributions, and how they frame the mathematics that is to be learned. I will argue that the second lesson, rather than the first contains seeds of more substantial learner-centred teaching, because of the teacher's clear focus on regulating and modelling the mathematics.

The data

The data comes from a larger research project (Adler, Lelliott and Slonimksy et al, 1997), which includes mathematics, science and English teachers in South Africa. In Brodie (1998), I present an analysis of a video-recorded lesson of each of the five secondary mathematics teachers in the sample. Categories for analysing the verbal interaction in the videos were developed according to the principles of networking (Bliss et al, 1983), and were generated both from the data and from the literature in classroom interaction. A more detailed description of the categories and results can be found in Brodie (1998). In this paper I will use some pertinent results from the analysis of two lessons to illustrate different possibilities for learner-centred teaching.

Lesson 1 is in a Grade 12 class, in a relatively well-resourced, urban school in Gauteng, with 24 pupils present during the lesson. Lesson 2 is in a Grade 11 class, in a more rural, less well-resourced school in the Northern Province, with 40 pupils present. The two lessons are similar in that standard tasks were set, one in linear programming the other in trigonometry, and the teachers interact with the pupils to work through the tasks on the board. Both lessons are entirely in the Initiation-Response-Feedback (I-R-F) form of classroom discourse first identified in British classrooms by Sinclair and Coulthard (1975) and elaborated by Edwards and Mercer (1987). However, a closer analysis of how the two teachers give feedback to pupils' responses suggests that the teachers' initiation and feedback moves perform different functions in these two classrooms.

Lesson 1 - Questions for "discovery"

In this lesson, the teacher interacts in ways that resonate with Edwards' and Mercer's (1987) descriptions. He asks questions to which he expects particular answers, and if the expected response is not forthcoming, repeats his prompts or questions until it is obtained. Edwards and Mercer argue that such question and answer sessions reflect attempts by teachers to allow children to "discover" ideas for themselves, as encouraged by the child-centred movement. However, in doing this, teachers find themselves in a paradoxical position, because they cannot allow pupils to discover just anything. Rather, they have to "funnel" (Bauersfeld, 1980) pupils towards the ideas they want them to discover. An unintended effect is that pupils resort to "guessing games" (Edwards and Mercer, 1987:34), since they have no means of establishing what is required for a successful response.

For example, in the following extract, the class has drawn the graph of a linear programming problem and shaded in the feasible region. They have provided a continuous shading and the teacher wants the pupils to come up with the point that only whole numbers should be included in the feasible region, requiring a discrete rather than a continuous shading:

T: *Look you've got everything here* [points to feasible region]. *Is this possible?*
P1: *Yes*
T: *Why do you say so?*
P1: *The amount (inaudible)*
T: *.... but now the way that you know, its not quite correct*
P2: *Sir, you say its not quite correct, you mean the lines?*
T: *Ja, I mean the whole thing, cause you see here you cannot do that, ne, its not quite correct ... Remind me what we are dealing with.*

[a pupil comes up to the board, and shades the feasible region with parallel lines instead of crossed lines]

T: *Its not 100% perfect. Now what are we dealing with here.*
P3: *Aircraft*
T: *Aircraft and?*
P4: *Passengers*
T: *So now can you have ½ a craft?*
Ps: *No*
T: *And can you have ½ a passenger? So everything, including fractions are included. So what you must understand is when we're dealing with persons or aircraft, we're dealing with the 'whole'.*

The teacher prompts the pupils by saying its not 'perfect', or not 'quite correct'. The pupils do not know what he is referring to, and therefore suggest the exchange of one kind of continuous shading for another. Because he does not provide them with a better frame for thinking about the nature of possible solutions to the problem, in the end, the teacher is forced to tell the pupils what he wanted to hear, that they are dealing with whole numbers. The rest of this conversation confirms that the pupils are still are not sure what the object of the discussion is:

T: *What kind of numbers are here? [points to the axes which show even numbers]*
Ps: *Even numbers*
T: *(laughs) Yes I know they're even numbers, because I couldn't start from 1. But from 1, 1,2,3,4?*
Ps: *Natural numbers*
T: *Alright*

The pupils' gaze is on the representation of the problem on the board, the numbers visible on the axes and the shading of the feasible region. They provide an acceptable answer to the teacher's question "what kind of numbers are here?" based on what they can see on the board. After this response, the teacher basically gives them the answer. He does not engage with their ways of thinking about what they see, and he does not formulate questions or tasks which might enable them to think about the solutions to the problem.

This lesson had a high level of pupil participation (see Brodie, 1998) and the pupils were clearly comfortable with answering questions. In an interview, the teacher explicitly talks about his approach as "the question and answer" approach. Asking questions is a possible interpretation of learner-centred teaching, as it allows for participation and involves the pupils in the classroom narrative. However, although this teacher has managed to allow pupil contributions, he does not provide a frame for useful contributions. His questions put the pupils in the position of having to work out what he is thinking, rather than developing their own thinking. When they do express their own ideas, he does not engage with them, possibly because he does not understand how they are thinking. Although increased participation by pupils might be seen to be learner-centred, on deeper analysis it appears to be superficial, particularly from the point of view of mediating the mathematics and the pupils' thinking.

Lesson 2 - Regulating and modelling the discourse

The second teacher, whose lesson is also in the I-R-F form, works somewhat differently from the first. There are fewer incorrect responses in the lesson, and to these she is more likely to explain why particular responses are incorrect, or to give the correct response, from her perspective (Brodie, 1998). Therefore it would seem that her questions and statements perform a different, or additional function from that of generating pupil participation.

The problem being worked on is to simplify $(\sin\theta + \cos\theta)/(\sec\theta + \operatorname{cosec}\theta)$ if $\sin\theta = -3/5$ and $0° < \theta < 270°$. The lesson can be divided into episodes which reflect the important parts of the problem-solving process: 1. determine in which quadrants sine is negative; 2. determine which quadrant θ is in; 3. determine x by using a diagram and the theorem of Pythagoras; 4. determine Cos θ and simplify the expression. The teacher's questions and statements work to distinguish and connect the different parts of the procedure. For example, she initiates episode 1 with the statement: *"firstly we must determine our sign"*, which is a general statement of procedure and focuses the pupils' attention onto the beginning of the process. During episode 3 she asks the question: *"in order to find cos*θ *what is the definition?"* which provides a rationale for finding x, and connects episode 3 to episode 4 and the ultimate goal of solving the problem[3].

These utterances serve to focus pupils' attention on the salient parts of the mathematics, and to provide links between the different parts of the problem-solving process, and rationales for what is being done. They function as a regulatory mechanism which guides the problem-solving process. Mathematicians solving similar problems might ask themselves similar questions. So this teacher is modelling a problem-solving process. In Bruner's (1985) terms, she is acting as "vicarious consciousness" for the pupils.

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[3] Space limitations do not permit extended transcripts from this teacher's lesson. These will be provided and discussed during the presentation.

This teacher's questions are mainly "closed" and usually have very clear "right" answers. The form and nature of her questioning allow her to maintain tight control over the discussion and knowledge generated in the classroom. On the other hand, the questions are clearly framed, and the pupils can answer them on the basis of the mathematics that they know. They do not need to try to work out what the teacher is thinking from clues that are given (which is a less obvious, but equally controlling form of discourse). For example:

T *What is the definition of sine?*

P1 *Opposite over hypotenuse*

T *Opposite over hypotenuse, which means if this is our theta, its the opposite of theta over the hypotenuse, so let's take this as y, because its parallel to the ...*

Ps *y axis*

T *And this is the hypotenuse which is r, and this is the x-axis, so which means opposite over hypotenuse which is y over r*

Modelling the use of questions as a regulatory mechanism in problem solving can be a powerful mediational technique in mathematics. A Vygotskian approach to learning and development suggests that a teacher modelling the process can be a first step in the development of self-regulation on the part of the pupils. What this teacher does not do (in this lesson), is to begin to hand over the questioning process to the pupils. She does not begin to remove the scaffolding, and so the problem solving process remains largely centred around her. The pupils responses remain at the level of 'gap fills' and recalling simple facts. Similarly to the first teacher, she does not attempt to find out what pupils are thinking, nor why they think in the way that they do. In contrast to the first teacher, she does mediate the mathematics.

When this teacher was asked what she thought about learner-centred approaches she answered that she did not understand the question. This suggests that she had not heard about learner-centredness and its concomitant "discovery" approach, and so approached her teaching as finding ways to best enable pupils to solve problems, which includes modelling what for her is a successful approach.This teacher can develop her approach to enable more possibilities for appropriation on the part of the pupils, or to enable other methods to be developed and discussed. However, if she were to lose the mathematical regulation that she provides for pupils in favour of some kind of "discovery", her pupils might lose out on important learning opportunities.

Conclusions

In this paper I have compared two lessons which display a similar form (I-R-F) but where the teachers' inputs, primarily their questions, perform different functions. I have argued that one lesson might look more like a learner-centred classroom, in the "discovery" sense. However, the teacher's questions do not serve to frame or enable the pupils' responses and the pupils are unable to make sense of what is required. The second lesson is more teacher controlled, but the teacher's questions serve to regulate and model the mathematics for the pupils. This lesson may have seeds for more learner- and mathematics-centred approaches, and therefore, for better mathematics learning.

My analysis suggests that firstly, for the idea of learner-centred teaching to be useful for mathematics teachers in South Africa, it needs to be clarified with and by South African teachers, taking their perspectives, practices and contexts into account. Learner-centredness is a complex notion and the relationships between learner-centred teaching and learning are not obvious. As a slogan, without clarification, it may be more of a hindrance than a help. Secondly, in order to be useful, the concept should not be put up in opposition to teacher-centred or mathematics-centred. The interrelationship between the three should be clarified in ways that do not leave teachers in contradictory positions because they are trying to teach mathematics, or such that they lose sight of their own roles and the mathematics in trying to acknowledge and accommodate learners.

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RESEARCHING TRANSITION IN MATHEMATICAL LEARNING

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A theoretical perspective on the ways in which children progress in learning mathematics and how this is treated in the research literature is examined. It suggests that there is a difficulty in associating teaching discourses with the mathematics they locate which can result in an incommensurability of alternative perspectives being offered. This resists attempts to privilege any particular account but rather demands an analysis of these discourses and their presuppositions. In particular, it argues that the shift in the student's mathematical development from arithmetic to algebra can be read in a number of ways and that alternative approaches suit different and perhaps conflicting outcomes such as demonstrating awareness of generality or performing well in a diagnostic test featuring the solution of linear equations.

This paper is about development in student mathematical performance, but not so much how to achieve it but rather how it is seen and how it functions as a notion in guiding our actions both in how we describe students progressing through successive stages on a developmentally formulated curriculum and in creating alternative approaches which better enable us as teachers to facilitate this. We shall review some work describing the transition children make from work in arithmetic towards simple algebra as an example of how student mathematical development is treated in the literature. We will examine how this development is understood within these studies and the ways in which they conceptualise the research task. We suggest the studies downplay the social construction of the students and of the mathematics they perform with a consequent suppression of the differences between the alternative mathematical projects that may present within any given educational enterprise.

THE TRANSITION FROM ARITHMETICAL TO ALGEBRAIC THINKING

We wish to focus on two key studies which address the student's transition from arithmetic to algebra: Filloy and Rojano's (1989) work which sees this transition as mathematically defined and Herscovics and Linchevski (1994) work which sees it as cognitive. These will provide examples of the different ways in which this transition is described within the research literature. We will also be referring to two studies by Sfard and Linchevski (1994 a b) whose work suggests an interesting disruption to any dichotomising of these two perspectives.

Didactic cut

How might we characterise the shift from arithmetic to algebra? Filloy and Rojano (1989) introduce the notion of "didactic cut" between arithmetic and algebra. They see this as arising when the child's arithmetical resources break down in tackling linear equations. They suggest that a sharp delineation between arithmetic and algebra can be identified when in a first degree equation we have the unknown on both sides, e.g. $Ax+B=Cx+D$. When the unknown only appears on one side they suggest the solution can be found intuitively through purely arithmetical means and hence with existing skills, such as counting procedures or inverse operation. Thereafter they claim additional resources are needed and to overcome this barrier the students require assistance from the teacher who needs to provide some sort of device which enables the student to negotiate access to the new domain. They go yet further by suggesting that this introduction of teaching strategies results in an inevitable diversion in reaching mathematical objectives. This is because the teaching devices create obstacles through their introduction of intermediate codes, ie between functioning at the concrete level and the fully syntactic algebraic level. These

> *"hinder the abstraction of the operations performed at the concrete level and are due to a lack, in the transition period, of adequate means of representing to which the various operations lead. The obstacles arise from a sort of "essential insufficiency' in the sense that modelling ... tends to hide what it is meant to teach.*

Such modelling, they suggest, is characterised by two components, namely transition and separation. They argue:

> *When either of these two components is strengthened at the expense of the other, the new objects and operations become harder to see*

For example, the noting of generality might become obscured if the student becomes locked within the domain of a particular model or teaching device, having separated herself from the task of seeking the abstraction implied by the more concrete domain.

Cognitive gap

Herscovics and Linchevski (1994, pp. 59-61, see also Linchevski and Herscovics, 1996) also seek a "clear-cut demarcation between arithmetic and algebra" but question the notion of didactical cut on the grounds, they claim, that it focuses on mathematical form rather than process. They introduce the notion of a "cognitive gap" which "is characterized by *the students inability to operate with or on the unknown*" (p. 75, their emphasis). This they see as moving the boundary being considered from one between two mathematically defined domains to one separating developmental stages in the learner's conceptions. Their findings with seventh grade students suggested that in equations where the variable appeared just once (e.g. $ax+b=c$ or $37-n=19$) nearly all students solved the equations arithmetically by inverse operations. However, a fundamental shift was noted when the variable appeared twice, either on just one side (e.g $ax+bx=cx+d$) or on both sides (e.g. $ax+b=cx+d$).

They found

> *the majority of students were able to solve them only by reverting to a process of systematic approximations based on numerical substitution. Although students managed to spontaneously group terms that were purely numeric, at no time did we witness any systematic attempt to group the terms in the unknown. We came to the conclusion that the students could not operate spontaneously with or on the unknown. The literal symbol was being viewed as a static position, and an operational aspect entered only when the letter was replaced by a number. This inability to spontaneously operate on or with the unknown constitutes a cognitive obstacle that could be considered a gap between arithmetic and algebra.* (Linchevski and Herscovics, 1996, p. 41)

Having identified this gap the authors then carried out empirical research to examine ways in which it might be crossed. Their findings include some suggestions of specific teaching techniques designed to overcome this particular gap, such as exercises in grouping like terms, developments of the balance model and decomposing into a difference to facilitate cancelling subtracted terms (Linchevski and Herscovics, 1996). They do however stress that "it is only when they (the students) achieve a more general perspective on equations, solutions and solution procedures that they can appreciate the value of a more general solution process" (op cit, p. 63)

Whether we privilege cognitive gap or didactic cut we suggest that the implication in such a demarcation of a "before" and an "after" present in both studies creates analytical problems in that it posits a singular subject progressing from A to B, in this case from the domain of arithmetic to the domain of algebra. We suggest that this creates difficulties in that it sets up a research model where we define a current actual state (functioning in arithmetic) and a future state we desire to attain (functioning in both arithmetic and algebra). Whilst we support Herscovics and Linchevski's decision to incorporate a concern for process we feel their conception of process is potentially too restrictive since they deemphasise processal dimensions rooted in the social constitution of both learner and the mathematics itself. As many recent research reports in mathematics education research remind us, mathematics and the students studying it are socially construed entities susceptible to temporal and compositional shift. We can always revisit and reread an interpretation of how a child proceeded through some mathematical exercises. We thus need to be cautious in defining states of mind, or even mathematical competencies, that position students in one domain or another. For this reason there is a need to be cautious in introducing models and to be more aware of their presuppositions and limitations. Further, conceptions of mathematics by the child generated through evolving understanding within sequences of exercises, discussions with teachers and peers, periods of reflection, periods of sleep, periods of forgetting, prevent stable characterisations of where the child is now in terms of developmental stages. Such critical perspectives have surely lost some of their appeal in any case after the work of contemporary writers (e.g. Walkerdine) who have resisted psychologically oriented accounts centring analysis around singular developing minds, such as that provided by Piaget's stage analysis. At best accounts of shifts by a child from arithmetic to algebra are held in place by analysis of quantitative data demonstrating child's facility in particular environments. We can, however, always create new tasks which simultaneously seek to capture whilst extending the domain we are examining and thus add to the evaluative devices we might offer in respect of it. We suggest that we cannot create a static picture without suppressing significant aspects of the processal features which give rise to the outcomes we are assessing. Also, the idea of "transferability" of mathematical learning to other mathematical or non-mathematical domains has proved to be a somewhat elusive concept in recent work on situated cognition (Lave, 1996), which further disrupts any attempt at specifying particular mathematical competencies or stages of development .

It also seems important to incorporate within any account a recognition of the storying carried out by the students themselves and the ways in which they situate any developing understanding of particular mathematical ideas within their broader conceptions of what constitutes mathematics (see for example, Ruthven and Coe, 1994, Rodd, 1993). Any addition to the student's mathematical repertoire is understood within a broader narrative frame within which narrower conceptions of mathematics reside. That is, students utilise a broad range of metaphorical apparatus in supporting their own mathematical thinking, situated within their broader narrative accounts of why things are as they are and how they connect with other bits of mathematics and life outside. The student's experience we conjecture is not of a straightforward switch from arithmetic to algebra, their storying backdrop needs to be extended at the same time, although as Sfard and Linchevski (1994 a) indicate children do this in different ways and have different needs as regards providing supporting rationale for their implementation of procedures. They show us that we cannot assume consistency between children as to their apparent readiness to occupy a new domain and that this readiness is not straightforwardly associated with broader mathematical ability. They discuss a child whose preference for considered interpretive assessment of meaning slows him down against a peer more amenable to unreflective implementation of techniques. It is this example which gives rise to their distinction between "interpreter" and "doer". They characterise the students' respective motives as follows:

"Teachers and researchers are often bitterly disappointed to find out that even the most reasonable and carefully implemented didactic ideas would not bring much change. In particular, they are frustrated by their inability to significantly improve students' understanding of mathematics. It seems however, that the meaningfulness (or should we say meaninglessness?) of the learning is, to a great extent, a function of student's expectations and aims: true interpreters will struggle for meaning whether we help them or not, whereas the doers will always rush to do things rather than think about them. The problem with the doers stems not so much from the fact that they are not able to find meaning as from their lack of urge to look for it. In a

sense they do not even bother about what it means to understand mathematics. (Sfard and Linchevski, 1994 a, p. 264)

This is interesting since it muddies the water in any attempt to draw clear distinctions between mathematical and cognitive domains. On the one hand we have unreflective performance of mathematical procedures, on the other a more sustained attempt to understand which seems to work against performance at least in the short term. This seems to disrupt any straightforward attempt to correlate cognitive ability with mathematical performance. If we accept Sfard and Linchevski's (1994 a) conjecture of there being at least two types of learners what consequences can we assume for the ways in which we construct learning theories and learners enacting them. The conflicting preferences of interpreter and doer each of whose mathematical progression is in different ways are dependent on, among other things, chosen teaching strategies, the assessment instruments applied and the learning theories used in explaining this progression. Thus learning theories might be seen as partisan, prejudicing against particular learners or against certain capacities or potentialities present within all learners. Insofar as mathematical learning supports both intrinsically mathematical concerns as well as more utilitarian enterprises, facility with both abstractions and concretisations seems crucial, where perhaps mathematical agendas privilege the former while more utilitarian agendas (including those frequently assumed within school mathematics) privilege the latter.

It was this work that led us to question the ways in which differences of results between the two key studies cited have been put down to sampling differences, different experimental conditions etc rather than to an overstretching of the research models being applied (Herscovics and Linchevski, 1994, p. 75). Distinctions between the respective intentions of interpreter and doer are not picked up by quantitative research focusing on facility with alternative algebraic forms. At the instrumental level of measuring algebraic achievement such distinctions are deemed irrelevant. But similarly, the very quest for some clearly stated boundary between arithmetic and algebra seems condemned from the outset since we cannot breach the inevitable divide that separates mathematical and cognitive domains within such models. Between the two models there is a dichotomous choice between seeing the transition as separating, in the former, two distinctive mathematical forms and, in the latter, two developmental cognitive stages.

MULTIPLE ACCOUNTS

Filloy and Rojano (1989, p. 24) talk in terms of "the direction of what algebra is intended to achieve", a direction which cannot be specified directly but needs to be alluded to through teaching devices which, whilst assisting us in broaching new territory, inevitably draw us away a little from the conceptual understanding we seek. But we have suggested targets expressed in terms of desired trajectories are problematic insofar as we attempt to claim movement from state A to state B. We cannot easily define an event to have happened which effects such a transition, for within any such event there is the possibility of multiple accounts each implicating alternative phenomenological features. This is interesting placed alongside two other important studies concerned with algebra. Sfard and Linchevski (1994 b, p. 191) assert that algebraic symbols do not speak for themselves, rather, any algebraic expression can be read in a number of ways. For example, the expression $3(x+5)+1$ can be read as a computational process, a certain number, a function, a family of functions or merely as a string of symbols. Indeed, they claim, we can identify "an inherent process-object duality in the majority of mathematical concepts" (ibid). Meanwhile Gray and Tall (1994) introduce the term "procept" to identify this duality. They suggest that the "ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider schema". We take this schema they refer to as being mathematical but as we have suggested earlier we can assume an even broader focus. Even within a mathematical domain we can broaden out in different ways. For example, the two key studies referred to earlier privilege accounts which assume arithmetic precedes algebra, the direct opposite to that described in the work of Gattegno (e.g. 1974). Meanwhile, we take from Mason's (1996) work that the reductionism implicit in the transition being specified in the two studies cited as a shift in the form of equation as drawing attention away from the underlying principle of algebra being about the noting of generality. The mathematician's ideality is both located but also evaded within the research models we create in our analysis. This in itself will come as no great surprise; models are inevitably simplifications introduced to help us see structure (e.g. Linchevski and Williams, 1996, p. 266). It does however bring into question the virtue of any quest to privilege any particular model or any final declaration as to the mathematical content this locates. For example, all of the studies we have cited here seem to place emphasis on attaining a better understanding of mathematics unfettered by the clumsy techniques that get us there. An alternative objective may be effective performance in tests designed to facilitate international comparisons in mathematical achievement. We suggest it is formats of learning and assessment rather than purer notions of mathematical understanding which underpin the hard currency required to make such comparisons possible. In some ways it may be that the language of the mathematician has been corrupted in the public domain. But when discussing mathematics in the public domain we must ask whether it is the mathematician or the public which takes precedence?

It may be helpful to offer an another example of how pedagogical discourses condition the mathematics learnt through them. In his analysis of a teaching scheme Dowling (1996) found mathematics which was designed for less able students to be of a very different nature to that given to their more able peers. For any given topic the emphasis in the instruction varied. Insofar as this is true more generally this would seem to result in exclusion for the less able from the real business of mathematics as understood in more abstract terms, caught as they are in the discourse of "less able" mathematics. This analysis identifies at least two levels of mathematics each characterised by a discourse with associated styles of illustration, questioning etc. But clearly there are many such discourses operating in mathematics education and as with the distinctions between mathematics designed in the schemes for less or more able students, differences between these are swept over in many situations as a consequence of outcomes being seen primarily on a register of mathematical content, independent of the processes that lead to these. Nevertheless each of these discourses misses the mathematics it seeks to locate and is characterised by some sort of illustrative approach which simultaneously serves as a teaching device but, in line with the analysis of Filloy et al., draws us away from the mathematics. This, of course, is also true of the mathematics designed for the more able students following the scheme where situations are couched in more overtly mathematical form. But each of these discourses is predicated on some sort of mathematical objective whether this be tied down to performance in a specific discursive frame such as the solving of a linear equation or more transcendental mathematical claims such as abstraction, the noting of generality or intuition.

Mathematics education research is generally predicated on some notion of improvement, whether this be children progressing through a curriculum or teachers or researchers developing improved strategies for facilitating this. Part of our task here has been to problematise conceptions of moving from one domain to another. We suggest source and target domains and the transition between them each resist phenomenological accountability. In our main example the studies cited are seeking to pin down essential characteristics of functioning in arithmetic as distinct from algebra. However, the stressing of certain features results in an assertion of a particular view of mathematics as though this can be specified independently of broader learning objectives, such as developing intuition or doing well on a diagnostic test featuring linear equations.

Ricoeur (1988, p. 241) has argued that "temporality cannot be spoken of in the direct discourse of phenomenology, but rather requires the mediation of the indirect discourse of narration". Features of time, progress, development and shift are not constituted through agreeable criteria, but all depend on interpretations reflecting attitudes produced within history, ideology and auto-biography. Any movement to a new way of understanding can only be spoken "by means of the complex interplay between the metaphorical utterance and the rule-governed transgressions of the usual meanings of our words" (Ricoeur, 1984, p. xi). Ricoeur suggests that this moves beyond mere *seeing as*, but rather becomes "*being as* on the deepest ontological level" (ibid, my emphasis). We cannot be limited to interpretation at the level of immediate comprehension but need to include reflected upon application within performance where the student's storying backdrop is given time to settle. We can thus create a notion of mathematical understanding that is not merely defined in terms of mental states but also incorporates some notion of time. We suggest that the teaching devices such as the sort introduced by Filloy and Rojano (e.g. balance or geometric) can be understood as contributing to this necessary and inevitable temporal dimensions of the constitution of the ideas we seek to address in our teaching. Moreover, mathematics cannot be seen or understood independently of the cultural filters through which we receive it. Further, mathematics as mediated and articulated through teaching devices comprises an essential dimension of the mathematics being learnt, and should not be seen merely as a means to an end. Proficiency with concretisations is integral to the broader proficiency of moving between concrete and abstract domains, a proficiency which lies at the heart of mathematical endeavours (at least in school). Indeed, one might suggest that for many students and many teachers proficiency in concretisations forms the backbone and principal motivation of activity pursued within the classroom.

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METAPHOR AS A TOOL IN FACILITATING PRESERVICE TEACHER DEVELOPMENT IN MATHEMATICAL PROBLEM SOLVING

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This paper reports on a study that investigated the effect of using metaphor as a tool in facilitating reflection and creating awareness of the nature of problem solving and teaching to facilitate students' development of this process. Two groups of preservice elementary teachers were studied using a humanistic research approach. Data consisted of the participants' writing of their thinking resulting from their reflections on a series of activities on problem solving pedagogy. Analysis involved determining the nature of the metaphors used and what the participants considered important in characterizing problem solving and its teaching. The outcome revealed that "cued metaphors" provided a significantly meaningful way in helping the preservice teachers to extend and enhance their interpretations of the nature of problem solving and its teaching.

Introduction

Recognition of the importance of teacher education in facilitating effective implementation of current reform recommendations in mathematics education (e.g., NCTM, 1989, 1991) seems to be the stimulus to the growing number of studies on the mathematics teacher. Studies on preservice teachers have covered their content knowledge (Ball, 1990), pedagogical content knowledge (Even, 1993), beliefs (Gorman, 1991) and development (Simon, 1994). However, consistent with the current climate in mathematics education that is promoting a problem solving perspective to teaching and learning mathematics, studies involving problem solving have not explicitly considered it as a curriculum topic but as a pedagogical process. Thus, preservice teachers were either engaged in a problem solving process to learn mathematics (Lester and Mau, 1993) or looked at in terms of how they implemented problem solving in their teaching of mathematics (Cooney, 1985). One could conclude from Cooney's study that preservice teachers' "natural" inclination to problem solving, from years of focusing on an algorithmic approach to problem solving, is likely to conflict with the problem solving perspective of teaching and lead to unsuccessful implementation of it. Thus, special attention is necessary in teacher development in terms of preservice teachers' understanding of problem solving and its teaching beyond the traditional algorithmic approach.

This paper reports on a study of preservice teacher development in problem solving. The goal of the study was to determine the effect of using metaphor as a tool in facilitating reflection and creating awareness of the nature of the problem solving process in dealing with non-algorithmic, non-routine mathematical problems, and teaching to facilitate students' development of this process. In particular, it compared the nature of this awareness resulting from using a reflective process involving "cued metaphors" and one with "uncued metaphors" and considered the nature of the metaphors used.

Theoretical Perspective

The study is framed in the context of teacher learning based on reflection on action, thinking, and/or past experiences. Dewey (1963) and, to a greater extent, Schon (1983, 1987) have been responsible for significantly influencing this role of reflection in teacher education. Such reflection has been promoted as a way in which teachers construct the meanings and knowledge that guide their actions in the classroom (Schon 1983, 1987). It also leads to re-construction of beliefs or construction of new perspectives of how one's teaching could be. Thus, the importance of reflection in teacher education is usually linked to the relationship between teachers' beliefs about content, teaching and learning and their classroom behaviors. More generally, it is linked to creating awareness of belief or conceptual systems framing one's thinking and actions.

Reflection has, therefore, gained significant acceptance as a basis of teacher education (Bennett, 1998; Grimmett and Erikson, 1988; Halton and Smith, 1995). Its importance to teacher preparation programs is linked to findings that preservice teachers have well developed personal and practical theories regarding teaching. They come to teacher education programs with strong theories of teaching acquired during many years of being a student (Brookhart & Freeman, 1992). These theories have been shown to influence the way they approach teacher education and what they learn from it (Calderhead & Robson, 1991). In general, they tend to rely on their personal experiences as learners in constructing meaning for classroom events. Thus, reflection becomes a necessary process to establish awareness of their personal theories to facilitate growth.

The use of reflection in this study, then, is associated with developing preservice teachers' craft knowledge in relation to mathematical problem solving. It is assumed that merely solving problems, without intentional reflection on the process involved, may not be enough for one to understand the nature of problem solving (and consequently, its teaching), since one tends to lack awareness of particularities of the process while engaging in it solely for the purpose of getting an answer. In this regard, metaphor is being adopted as a tool to facilitate intentional reflection on problem solving and its teaching. This use of metaphor is based on the view of Lakoff and Johnson (1980) that it is a way in which one makes sense of one's world. Thus, it provides a conceptual framework for thinking about something and, consequently, shapes the way one thinks. It determines what one sees happening in a particular situation, the way one interprets an event, the solutions that are attempted, and the manner in which one is likely to behave. It communicates messages about the meaning one constructs. In terms of this view, metaphor has been used as a research tool to understand inservice teacher thinking (Grant, 1992) and to facilitate professional development of preservice teachers (Bullough, 1994; Marshall, 1990). The latter has focused on teaching metaphors, i.e., ways of conceptualizing teaching, with no particular focus on specific content areas. In this study, the focus is more related to pedagogical knowledge regarding one area of mathematics — problem solving. Thus, the use of metaphor is exploratory to obtain information on its worthiness in preservice mathematics teacher education.

Methodology

Methodology involved working with two groups of preservice elementary teachers in their final term of their undergraduate education degree. All of the participants completed at least 2 full courses of university mathematics, a half course of elementary mathematics methods (did not include a focus on problem solving), and all of their practice teaching. The study was conducted during a one-term, post-practicum year, elementary mathematics methods course they were required to take as elementary mathematics education majors. One group (8 females) was registered for the course in the winter of 1996 and the other (7 females) in the spring of 1996. Both courses were taught by the researcher and involved the same topics. About half of the course was spent on non-routine problem solving. The winter group worked with the uncued-metaphor approach (did not require a conscious determination of a metaphor) and the spring group with the cued-metaphor approach (required a conscious determination of a metaphor).

The common, core aspects of these approaches were: First, the student teachers solved non-routine problems individually, then reflected on this experience and past experiences solving such problems to determine their understanding of the nature of the problem solving process. Second, they reflected on their thinking of how to teach problem solving. Third, they reflected on readings on problem solving in mathematics focusing on similarities and differences to their views in the first two activities, and any significant revisions they would make to them. Four, they practiced with each other teacher intervention during students' problem solving, taking turns being teacher and student, and reflected on when, how and why they intervened.

For each activity, all reflections were done first on an individual level, then on a group level, then again on an individual level. This allowed for dialogical interactions in terms of self-self and self-other relationships. All levels of reflections were accompanied by written journals of each participant's thinking of each situation they examined. In terms of its meaning as a conceptual framework for thinking about something, metaphor was used by both groups of preservice teachers in their reflection of problem solving. Thus, "uncued-metaphor group" and "cued-metaphor group" are used instead of "non-metaphor group" and "metaphor group", respectively, to distinguish the two groups. The uncued-metaphor group was simply told to do the assigned reflection and write journals on their thinking in their own words. The cued-metaphor group was told to think of and describe a metaphor that reflected their experience of the problem solving process. The metaphor was to then be used as a basis of their reflections and journals for all of the activities. They were given no guidelines regarding the nature of the metaphor because it was one of the factors being investigated.

Since reflection is related to perception of reality, a humanistic perspective of research seemed to be most relevant to this study. In particular, the underlying rationale for the form of data and analysis was linked to phenomenography (Marton, 1988) which focuses on people's perceptions of a phenomenon. Thus, data for the study was all of the participants' journals associated with the problem solving activities. The journals were considered to be an indication of the nature of the participants' reflections, learnings and understandings. Analysis consisted of identifying the most "distinctive characteristics" (Marton, 1988) that appeared in the data regarding the effect of the two reflective approaches. The effect was considered in terms of the nature of the metaphors used and what the participants considered important in characterizing problem solving and its teaching.

Results

The results are presented in terms of the nature and effect of the two reflective approaches with emphasis given to the cued-metaphor approach. In general, the approaches resulted in significantly different quality of reflection.

Uncued-Metaphor Approach

The uncued metaphor used by the participants seemed to be a pre-existing framework unconsciously called into action to interpret the activities. It portrayed a frame consistent with the traditional, absolutist classroom view of problem solving. Thus, it made sense that it was used by all of the participants given their similar backgrounds with traditional mathematics classrooms. Through this frame, the participants focused on a decontextualized, algorithmic way of thinking about the problem solving process and the teaching process. They considered both of the processes in terms of "steps" to get to the outcome. The following example from one student is representative of how they viewed the problem solving process.

Read question through quickly, skimming.
Read question over carefully.
Write down useable information, draw a diagram, formula etc.
Develop the strategy further and use to find the solution.

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> *Find answer.*
> *Look back at process, check calculations.*
> *Ask if the solution makes sense.*

In general, problem solving was depicted as a sequential process in which a successful solution depended on a clear, logical choice among alternative strategies. Consistent with this, the teaching of problem solving was viewed as guiding students through these steps. For both the problem solving and teaching processes, affective factors were ignored.

These participants were also more "paradigmatic" (Bruner, 1986) in their treatment of the readings on problem solving. They focused on "context-free and universal explications" (Bruner, 1986), thus isolating procedural features of the problem solving endeavor from meaning of experience. In general, they used the readings to justify their thinking while trivializing or ignoring other features or considering them to be different from their thinking only in form. On a test based only on the readings at the end of the course, most of their answers reflected their thinking prior to exposure to the readings.

Cued-Metaphor Approach

The nature of the reflection was significantly different with the use of the cued-metaphors. These participants' focus was more on a humanistic, holistic, contextual way of thinking about the problem solving process and the teaching process. This humanistic context was set by the nature of the metaphors they selected. Everyone had a different metaphor. The metaphors involved activities (the metaphor activities) based on their personal, real world experiences that had a strong emotional impact on them. They also expressed their personal experiences and feelings about problem solving. The following are examples of their metaphors for problem solving (i.e., problem solving is like ___) and what they identified as the key factors influencing their choices of them.

> Downhill skiing: "challenging and yet fun ... requires skill, practice, courage, and perseverance"
> Fishing: "patience and persistence"
> Eating broccoli (a vegetable): "unpleasant but beneficial"

Both the metaphor activity (e.g., downhill skiing) and the problem solving process were considered in terms of procedural and affective factors. The following is an excerpt from one student's writing comparing her metaphor and problem solving experiences:

> *...While in line and then on the chair I scan the ski runs on the mountain. It seems like such a long ride to get to the top. I then begin to feel a little anxious. My only means of getting back to the chalet is to ski down. I know I can do it, but I am a little nervous so I usually decide to stick with one of the easier runs.... Without a doubt, sometime throughout the day I fall as a result of a mogul or a dip in the snow. Seeing as I am a*

fairly persistent person. I always get back up and keep going, sometimes choosing a different route down. ... When I finally reach the bottom, I am relieved.... As the day goes by, my confidence increases and I eventually try out more challenging runs....
For the most part I approach problem solving quite eagerly as I do with skiing. ... I glance over the problem to get a general feel for it. After reading the problem the entire way through, I sometimes feel anxiety like I do when I look down the mountain before skiing my first run of the day. I take a step forward to assess the situation and begin to record some of the relevant information. I choose a strategy to try. ... As I work through the problem, there are often points at which I get stuck or stumble and do not know how to move forward toward the solution.... Oftentimes it takes re-examining the strategy that I have been using and choosing to follow a different path towards the solution. Just because a new strategy is chosen doesn't guarantee a smooth path. This process may have to be repeated several times before reaching the final answer.... When I finally reach the solution I feel like a conqreror!... My confidence in my ability to problem solve grows with each successful experience. ...

The problem solving process emerging from these metaphors involved a preparation stage, a decision making stage, an execution stage, anxiety/tension/uncertainty, connection to context, perseverance, disappointment, and satisfaction. Consistent with the problem solving process, the teaching process took into account the affective factors. The following excerpt of the description of the teaching of the metaphor activity (downhill skiing) illustrates their basis for considering the teaching of problem solving.

... I would then encourage the student to attempt to head down the hill at his/her own pace, trying to use the theory and techniques that he/she had been taught. ... I would offer hints if requested or if I noticed that she/he was having a lot of difficulty and experiencing frustration. ... As the student become increasingly comfortable with the sport, I would "stand back", let them make the decision as to the type of run, and allow them to fall and get back up again without interfering too much....

Building on her way of teaching her metaphor activity, each participant determined a similar basis of teaching problem solving. This emerged as consisting of a teacher-guided, skills and confidence development period and a teacher-facilitated, self-development period. The latter required a more hands off approach by the teacher with students engaging in more self-initiated and self-management processes. The specifics of the teaching process were dependent on the nature of the particular metaphor. However, the emphasis was on making students comfortable, easing them into the situation, providing guidance and support, and allowing them to experiment. Teacher intervention was also described in terms of metaphors, for example: *Intervention is like conducting a band....*

Like the uncued-metaphor group, these participants used the readings on problem solving to validate their thinking, but they also used them to obtain details regarding the

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nature of skills and strategies for problem solving and other factors dictated by the particular metaphor activity. They needed this detail to build a "story" of problem solving that paralleled that for the metaphor activity. In fact, the more details they described about these factors for the metaphor, the more details were drawn out about problem solving from the readings. Similarly, the way in which they described how to teach the metaphor activity, dictated the need for specific details of equivalent features in considering the teaching of problem solving. In general, these participants showed more adaptability to the readings. The cued metaphor seemed to facilitate better retention of the readings based on their responses to the same test given to the uncued-metaphor group.

Conclusion

The potential of cued metaphors as a vehicle for raising preservice teachers' pedagogical awareness of problem solving seems to be significant. The cued-metaphor group had similar background in terms of mathematical problem solving as the uncued-metaphor group. However, when asked to reflect on problem solving, the only construct for the latter seemed to be the traditional framework (the absolutist view), a frame that made them lose sight of the humanistic aspects of problem solving. But the cued metaphor enabled the other group to extend and enhance their interpretations of what they saw beyond this traditional view. It provided a more humanistic way for them to learn from experience and allowed them to hold knowledge differently and more meaningfully. It contributed to a different level of sense-making about the nature of problem solving.

In general, the metaphors, cued and uncued, provided a way for the preservice teachers to frame their understanding of problem solving. But these frames, in turn, suggested the ways they would likely develop their conceptions of teaching problem solving. Chapman (1997) argued that metaphor played a significant role in how experienced teachers organized and conducted their teaching of problem solving. These metaphors were shaped by the teachers' conceptualization of problem solving based on their personal experiences. Thus, using cued metaphors for preservice teachers could enhance their development in terms of the frame they construct to organize their teaching.

Metaphors do have limitations in that they create boundaries that may screen out salient information from one's awareness. The uncued metaphor lens was more restrictive than the cued metaphor lens. However, all cued metaphors selected by students may not be equally meaningful in facilitating their understanding of problem solving. Sharing of metaphors becomes important to broaden the boundaries of any one metaphor and to reconstruct it in more meaningful ways. A combination of the cued and uncued metaphor approaches could also more likely provide a more complete picture and understanding of problem solving. Cued metaphors can also be used as a way of sharing theory, as a basis for students to resonate with their own metaphors. The study also suggests that identification and analysis of metaphors is a promising avenue for uncovering and then exploring assumptions about the teaching of mathematical problem solving. Finally, problem solving itself can be conceived of as a metaphor for teaching and learning. Thus metaphors used to conceptualize problem solving could provide a basis of understanding teaching from a problem solving perspective. Such an approach could likely develop the kind of flexibility in teachers' thinking that allows them to realize reform recommendations in the teaching of mathematics.

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RESTRUCTURING CONCEPTUAL AND PROCEDURAL KNOWLEDGE FOR PROBLEM REPRESENTATION

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In this study I examine the question, what is the nature of prior geometry knowledge that would facilitate the construction of useful problem representations. The quality of prior knowledge is analysed in terms of schemas (Marshall, 1995). The frequency of schemas activated by high- and low-achieving students were compared under two conditions: problem context and non-problem context. The results showed that the two groups differed in the number of schemas activated during the course of solution attempt. However, in the non-problem context both the groups accessed approximately equal number of schemas. These results are interpreted as suggesting that the schemas of the high-achievers are more relevant for the negotiation of the knowledge states in the problem space, thus helping these students construct representations that have the potential to lead them to the solution.

Representational studies of mathematical problem solving emerged from concern with students' difficulties with problem comprehension, and what role, if any, does the structure of content knowledge play in the construction of a particular representation. The last decade has witnessed considerable investments in two fundamental aspects of problem understanding and representation: nature of prior knowledge and use of that knowledge during problem representation (National Council of Teachers of Mathematics, 1989). It is suggested that effective use of prior topic knowledge during problem solving is dependent upon the organisation of that knowledge (Prawat, 1989).

Knowledge organisation and mathematical activity

There is a growing body of evidence to support the view that qualitative aspects of students' content knowledge could exert a major influence on the deployment of the prior knowledge during problem solving. Quality of mathematical content knowledge is interpreted in terms of the degree of organisation of the different bits of mathematical information. Network models of knowledge organisation (Rumelhart & Ortony, 1977) provide a useful framework in which to visualise how mathematical knowledge is organised. A well-organised knowledge can be seen as one which has many components that are built around one or more core ideas. There are connections between the core concepts and the components, and among the components. The components could comprise mathematical definitions and rules as well as knowledge about how to deal with a class of problems. That is, organised mathematical knowledge encompass both declarative and procedural knowledge (Anderson, 1995).

The issue of organised content knowledge in the human memory has led to the development of a key psychological framework called *schemas* (Marshall, 1995). According to Marshall, schema is a cluster of organised knowledge that help students understand and represent a given problem, and provide cues for the activation of relevant strategies during the solution process. An important characteristic of schema is that they control students' processing activities by identifying the relevant aspects of the problem. This point was made by Mayer (1992) who suggested that both schematic and strategic knowledge need to be activated in any successful mathematical problem-solving effort.

Schemas have also featured prominently in studies of experts vs novice comparisons. In a study involving sorting problems, Chi, Feltovich and Glaser (1981) found that experts used schemas that were elaborate and contained principles underlying the problems whereas novice's merely attended to superficial features of the problem. These results led Chi et al to conclude that qualitative differences in prior content knowledge could explain why novices respond to the 'surface structure' of a problem while experts respond to its 'deep structure'. Similarly, Owen and Sweller (1985) pursued the question of importance of organised content knowledge in their study of trigonometry. The results of this study showed that students who produced correct solutions in the least amount of time tended to access and use previously acquired schemas that were structured around properties of right angles and other figures as well as knowledge about how to deal with problems involving right-angles, i.e., students invoked schematised knowledge of trigonometry.

The foregoing analysis and results suggests that successful students utilise mathematical schemas during problem solving. The study of the relationship between these structures and the outcome of a solution attempt constitutes an important area of investigation. The study reported here takes up this question.

Schema activation and problem representation

Building a problem representation is a complex process in which students attempt to establish meaningful links between elements in the problem statement and knowledge embedded in their schemas about that problem. In their analysis of problem understanding, Hayes and Simon (1977) have suggested that 'the representation of the problem must include the initial conditions of the problem, its goal, and the operators for reaching the goal from the initial state' (p.21). Thus, representation requires that connections are made between elements of what is given in the problem with components that are present in the relevant schema that is accessed. It follows that the more elaborate a schema is the greater the likelihood that students will be able to a) construct correct or useful representations and b) construct multiple representations of the same problem. It therefore, appears that the richness of the problem schema plays a pivotal role in helping students filter irrelevant information from given information and attend to information that would be relevant to working out the solution.

A second area in which schematised mathematical knowledge can play a significant role in directing problem-solving processes is mapping, a strategy in which the solver attempts to establish correspondence between the features and relations in the known problem (base) with those of the unknown problem (target problem). A successful mapping procedure requires that students go beyond the superficial aspects of the base problem in order extract its structure as encapsulated by key features and relations, and use that structure to solve a new problem with a similar structure. Information processing during mapping demands that students draw out the similarities between base and target problems something experts would do more effectively and rapidly because their processing of problem structure is driven by sophisticated and powerful schemas than those of novice problem solvers.

The function of schemas in modelling problems was investigated by Chinnappan (in press). The focus of this study was to examine the relationship between schemas activated by students and how these schemas were deployed during the construction of mental models. The results of this study revealed that a) students accessed a range of schemas relevant to the problem, b) the successful students were able to align components of the schemas in ways that suggested an understanding of the problem structure and c) high-achieving students tended to build more complex mental models for the problem resulting in novel paths to the solution than low-achievers.

Thus, schema-driven problem search and representation constitutes an important issue in our understanding of mathematical learning and problem solving. The identification and probing of schemas that students activate in relation to the solution of a specific problem has the potential to provide insight into the type of schemas that students develop in the problem area and how they harness them in constructing representation of problems. Both these issues are taken up in this study.

One could adopt two strategies to generate data relevant to the issue of schema development and problem solving. The first approach could analyse schemas that are activated by students in a problem-solving context. In this context schema accessing can be argued to be controlled by the need to achieve a goal, i.e., the solution of the problem. Secondly, one could provide a task which has most of the basic elements of the above problem but the students are not required to solve it. This latter approach is important as it could inform us about the schemas that students activate or fail to activate which would allow them understand the deep structure of the problem. The assumption here being that releasing the student from the constraints of solving the problem could facilitate the accessing of a greater range of related schemas, thereby, making more cognitive resources available for problem representation.

Method

Participants

Thirty students from five Year 10 mathematics classes in a middle-class suburban high school volunteered to participate in the study. The mathematics classes at this school were ranked on the basis of students' mathematics performance in the previous years. Class rankings were relevant to the purposes of the present study because they provided a useful way to identify students with different levels of geometry knowledge schemas the assumption being that students from the top-ranked class would have developed more elaborate and sophisticated schemas than those from the lower-ranked classes. The high-achieving group comprising fifteen students came from the top-ranked year 10 class, while the fifteen low-achievers came from the bottom two year 10 classes.

Tasks, Materials and Procedure

The purpose of the present study was to identify geometry schemas that students have acquired and to describe how they use these schemas to understand the structure of a given problem. This was achieved by developing two tasks which were different but related in terms of underlying geometry schemas. The first task was a plane geometry problem (PGP) which included a statement and a diagram. Students were required to find the length of an unknown segment. This task provided the problem-context in which to view schemas. The second task of the study involved students working on the diagram from PGP without having to solve any aspect of the diagram. This activity was considered to be appropriate for the observation of geometry schemas in a non-problem context.

Upon the completion of the problem, students were asked to work on the second task. Two sets of instructions were given for this task. Firstly, students were asked to study the figure, and a) identify all geometric forms that they could recognise and b) state any theorems, rules or formulae which they would associate with each of the forms they were able to recognise. In the second set of instructions students were asked to expand the figure in any way they wished, after which they were required to identify new forms and associated theorems that were created as a result of additions and modifications to the original figure. All students' responses were video recorded and transcribed.

Results

The store of students' schematised knowledge of geometry was analysed by determining the frequency of the activation of these structures under two contexts: problem and non-problem. In the first context, a frequency count was made of schemas that were used by high- and low-achieving students during their solution attempts of the problem. In Table 1 this is referred to as 'Problem Context'. Schema activation in the non-problem context was considered under four categories. The first category, labelled as 'Diagram Intact (open-ended)', contains schemas accessed by students whilst they were analysing the diagram. The 'Diagram Intact (problem-relevant)' shows schemas from 'Diagram Intact (open-ended)' that were relevant to a correct representation of the PGP. The third category, 'Diagram Extended (open-ended), consisted of geometry schemas that were activated as a consequence of expanding the diagram. And finally the 'Diagram Extended (problem-relevant) category shows the number of schemas from 'Diagram Extended (open-ended)' that were relevant to the solution of PGP. Table 1 also shows a fifth category, 'Total (problem relevant)', which is essentially the total number of problem-relevant schemas that were activated by the two groups of students in the non-problem context.

Table 1: Total number of schemas under problem and non-problem contexts

Context of Schema Access	Low-Achievers (n=15)	High-Achievers (n=15)
Problem Context		
Solution of problem	52	108
Non-problem context		
Diagram Intact (open-ended)	162	179
Diagram Intact (problem-relevant)	38	60
Diagram Extended (open-ended)	67	39
Diagram Extended (problem-relevant)	18	18
Total (problem-relevant)	56	78

Table 1 shows that regardless of the context high-achieving students tended to activate greater number of geometry schemas than their peers in the low-achieving group. This pattern is evident in all categories of the two contexts except one in which students were given freedom to extend the given figure, and the schemas were not relevant to the problem in question (Diagram Extended - open-ended). As expected when problem-relevant schemas were considered the high-achievers activated greater number of these knowledge structures than the low-achieving students. The difference between the groups is greatest in two categories: Solution of Problem and Diagram Intact (Problem Relevant). Interestingly, both these categories involve schemas that are relevant to solving PGP.

Comparison of schemas that were not directly relevant to the solution of PGP shows a lack of appreciable difference between the groups in two categories of analysis: Diagram Intact (open-ended) and Diagram Extended (open-ended). In fact, contrary to expectations, the low-achieving students generated greater number of schemas than the high-achieving students when they were required to expand the diagram and identify schemas.

Discussion

The aim of this study was to investigate the nature of schemas that students developed in the area of Euclidean geometry. Specifically, I sought to learn more about the quality of these domain-specific knowledge structures by examining schemas accessed by students in two contexts - problem and non-problem, and explore the interrelations among these pieces of knowledge. Two hypotheses were central to the aims of the study. Firstly, it was predicted that during the solution attempt high-achieving students would activate and use a higher number of *problem-relevant* schemas than low-achievers. Secondly, it was hypothesised that high-achieving students would also activate greater number of schemas than the low-achievers in a non-problem situations.

Frequency analysis of schemas provided support for the prediction that students in the high-achieving group activate a larger number of these knowledge forms than their peers who were considered to be low-achievers when the students were asked to solve a problem, or extend diagram that was similar to one that appeared in the problem. This result is consistent with other studies of mathematical knowledge structuring and problem solving that were concerned with the magnitude and quality of mathematical knowledge base and problem search processes (Prawat, 1989). Shoenfeld (1987), in his analysis of geometry problem solving, showed that good students tend to not only build larger networks of mathematical knowledge than those who are not as good but more importantly, this store of knowledge is better organised. The better structured knowledge base of the high-achievers of this study appear to drive moves during their solution attempts. Newell (1990) drew attention to two types of search in the problem space - problem search and knowledge search, both facilitated by a rich of store of schematised domain knowledge of the type built up by the high-achievers here.

The results of this study, however, did not support the hypothesis that high-achieving students would access greater number of schemas than the low-achievers in a non-problem context. Contrary to expectation, the amount of knowledge from both the groups in non-problem contexts was almost equal with low-achievers doing better in one task that was concerned with extending the diagram and exploring new schemas. In the non-problem contexts, students were given the figure that appeared in the problem context, and were required to analyse the figure for schemas. Further, students were invited to draw on the figure as a way to expand it in any number of ways. A key feature of this task was the absence of a problem goal, and there was considerable opportunity for students to experiment with the figure. The expectation was that the high-achievers, given their larger knowledge base of geometry and related concepts would do more with the figure, and consequently would activate more schemas. This was not the case, however. The results of frequency count suggest that low-achieving student do build up a considerable amount of geometric knowledge just as their high-achieving counterparts.

Taken together, the tentative support for first hypothesis and lack of support for the second hypothesis provides interesting insight into the nature of geometric schemas that are constructed by high- and low-achieving students. In non-problem contexts, ability level does not seem to have a significant effect on the knowledge accessed in the area of deductive geometry. However, when a problem-solving condition is imposed, the high-achieving students tend to activate more relevant schemas than the low-achievers. One possible explanation for this difference is that schemas constructed by the high-achievers are qualitatively superior. That is, these students are able to build multiple links between new geometric information and information that is already stored in their memory. For example, when teacher discusses the theorem that the diameter of a circle subtends an angle of ninety degrees at the circumference, students are generally given the figure or asked to deduce that the above theorem creates a right-angled triangle in a semicircle. Because the high-achievers have built up more conceptual points in their repertoire of mathematical knowledge, these students can now be expected to examine this information and create more meaningful links than the low-achievers. They could further invoke their prior knowledge about Pythagoras' theorem and trigonometric ratios and explore potential problems that could arise in a semicircle or they could link this theorem with other related theorems such as angles subtended at the centre of the circle is twice that subtended at the circumference. As students build these relations, over a period of time, knowledge built around the core idea of right-angles in a semicircle spreads in numerous meaningful directions. Anderson (1995) referred to this spread in knowledge network as an important mechanism in building domain knowledge.

The more powerful and better-connected schemas exhibited by the high-achievers play a vital role in facilitating understanding of problem structure. In the present study, the high-achievers were more adept at decoding the structure of the problem as reflected in the greater number of correct solution outcomes produced by these students. In contrast, schemas from the long-term memory of the low-achievers were less sophisticated and integrated and therefore, less effective in decomposing the problem in ways that would reveal its structure. As a consequence, these students activated fewer of these relevant schemas in conditions which demanded accessing and search for solution of the problem.

The richness of schemas that students of both ability levels activated when they were required to expand the figure and identify associated theorems and formulae suggest that students' geometric knowledge is integrated to some degree. However, this level of integration appear to be insufficient when the task demands that schemas be utilised in uncovering the structure of the problem. Teaching of geometry has to explore ways of facilitating the construction of more complex elaboration of ideas. That is we need to devise learning environments that has the potential to 'help students develop knowledge structures would make structural relations more salient' (Bassock, 1990: 532). Such schemas would facilitate the transfer of prior geometric knowledge to the representation and solution of novel problems.

In this present study I attempted to investigate the conceptual structure of geometric knowledge by examining the type of schemas students were able to activate in the context of a particular problem and its variant. While there is some evidence here to support the claim that the quality of schematised geometry knowledge have an important effect on representation of problems, the results of this study are based on the analysis of one problem. There is, therefore, a need to replicate this study with a variety of geometry problems and examine connections among the associated schemas. The identification of the links among geometry schemas that would help low-achievers understand problems is an important area for future research.

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THE STRUCTURE OF STUDENTS' BELIEFS TOWARDS THE TEACHING OF MATHEMATICS: PROPOSING AND TESTING A STRUCTURAL MODEL

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The present study examines the structure of students' beliefs about mathematics teaching. It addresses two main questions: (a) What is the nature of students' belief systems about mathematics, and how are the components of belief systems related to each other? (b) Are there differences in the structure of students' beliefs in terms of gender and grade level? These questions are addressed by estimating a theoretically informed multivariate causal model using data from students in grades 6 and 9 as input to the EQS computer program.

In recent years, mathematics educators have focused attention on rethinking the process of mathematics education at all levels. Calls for reform of mathematics education now urge teachers and faculty to improve not only the cognitive side of instruction, but also to emphasize non-cognitive issues, such as students' feelings, attitudes, beliefs, interests, expectations, and motivations (NCTM,1989).

Beliefs are important concepts in understanding students' thought processes, practices, and change. Students' beliefs strongly affect what and how they learn and are also targets of change within the process of teaching and learning. Beliefs are thus thought to have two functions in mathematics learning: The first relates to the constructivist theories of learning that suggest that students bring beliefs that strongly influence what and how they learn (Torner, 1997). The second function relates to beliefs as the focus of change in the process of education (Pajares, 1992). In this paper, we outline a model for studying students' belief systems about the "teaching of mathematics", which may accommodate both functions of beliefs: the one that considers beliefs as central to learning mathematics and the other that relates beliefs to changed mathematical behavior. To this end, the definitions of the terms "beliefs" and "belief systems" are first examined, followed by a brief overview of the proposed theoretical model. To illustrate the model, the methodology of the study, the description of the latent constructs and analysis are then delineated. Lastly, the results and the conclusions are summarized.

Definitions and the Purpose of the Study

Schoenfeld (1994) defined beliefs as an individual's understandings and feelings that shape the way the individual conceptualizes and engages in mathematical behavior, while Pehkonen (1997) explained that beliefs constitute the subjective knowledge of mathematics. In this study, we combine the above definitions and thus by beliefs we mean one's subjective knowledge about self and mathematics.

The spectrum of an individual's beliefs is very wide, and forms a structure with multiple components, which influence each other. The construct "belief system" is a metaphor used to describe how one's structure of beliefs about mathematics is organized. The belief system consists of three components: cognition, affect, and action (Torner, 1997). The cognitive component can be considered as the subjective knowledge of mathematics, the affective component refers to the emotional relationship with mathematics, and the action component is relevant to the readiness or tendency of a person to act in a certain manner. Thus, beliefs towards mathematics constitute a very complex and multi-layered system that enables individuals to find orientation in their environment.

The present study is restricted to the cognitive level of beliefs, and emphasizes on the identification of the structure of students' beliefs and views towards the teaching of mathematics. A number of previous studies referred to belief systems but few of them have examined the relationships among the different aspects of mathematical beliefs. What is new in the present study is the identification of a theoretical model for describing the hierarchical connections of the components of students' beliefs. Specifically, the study purports to answer the following questions:

- What is the structure of students' belief systems about mathematics and how are the components of belief systems related to each other?
- Are there differences in the structure of students' beliefs in terms of gender and grade level?

The Proposed Theoretical Model

The model is informed by the theoretical tradition that views students' belief systems as consisting of their beliefs about the nature of mathematics, the learning and teaching of mathematics, as well as about students and teachers' roles during mathematics lessons (Pehkonen, 1997). In the present study, we argue that students' belief systems constitute a three level hierarchy as it is depicted in Figure 1.

At the first level of the hierarchy, students, in their daily involvement with mathematics, form their beliefs towards the content (F1), the nature (F2), the teaching (F3), and the learning of mathematics (F4). Students' beliefs towards the content and the nature of mathematics are interrelated and form a second order factor (F5). The latter factor reflects the interrelations between students' beliefs about the nature and the content of mathematics, and it therefore reflects students' "images of mathematics".

Students in their early school years become acquainted with the content of mathematics through numerous activities and problems assigned by their teachers. We assume that these activities and problems influence their motivation and interest in mathematics. Students, for example, who perceive mathematics as a useful subject or enjoy mathematics content are motivated to engage in more mathematical activities, and thus develop more positive beliefs about the nature of mathematics. On the other hand, how students perceive the nature of mathematics is an important determinant of their beliefs towards the content of mathematics. However, it is hypothesized that students' views about the content and the nature of mathematics do not influence their belief systems in a direct way. This influence goes through the second-order factor "images of mathematics", which represents the common variance of students' beliefs towards the content and the nature of mathematics. It is assumed that students with positive "images of mathematics", i.e., students who believe that mathematics is a school subject accessible to all students, consider mathematics content in its correct dimensions. On the other hand, students with negative "images", believe that mathematics is only for the talented individuals, and thus consider mathematics as an abstract structure of knowledge, which consists of accumulated rules, formulas, terms, and algorithms.

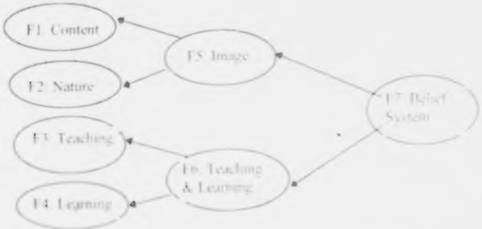

Fig. 1: The proposed theoretical model

A second-order factor (F6) is also formed by the interrelations of students' beliefs towards the teaching and learning of mathematics. Students' involvement in the learning of mathematics increases their interest and motivation which subsequently results in perceiving teachers' practices and teaching methods as more valuable in developing their abilities to understand mathematics.

It is also assumed that students develop beliefs about their own abilities to do mathematics in relation to their teachers' practices in the classroom. Students tend to perceive their own successes or failures and consequently the abilities needed to do mathematics from the various reactions of teachers reflected not only in grades but mostly through teachers' methods and expectations. These various forms, in turn, become the bases on which students judge themselves and thereby form their own concept or opinion of themselves as students and their own opinion of the specific abilities which are appropriate for the learning of mathematics. Thompson (1992) showed that the way students perceive the teacher's role in organizing instructional practices influences their beliefs and achievement. On the other hand, McLeod (1994) indicated that the pressure exercised on children to cope with highly demanding tasks, together with unimaginative instruction and non-positive teacher attitudes have destructive impact on their beliefs of mathematics. Thus, as in the case of content and nature of mathematics, the latent constructs "teaching" and "learning" affect students' belief system indirectly through the second-order factor "teaching and learning mathematics" (see Fig. 1).

Although different patterns of relationships can be argued theoretically, there are no empirical studies so far, to the best of our knowledge, which examine these relationships or their causal ordering. The model stated in the present study was that students' belief systems constitutes a hierarchy and can be decomposed into four first order and two second order factors. Specifically, it is assumed that students' belief systems are gradually influenced by students' initial views of mathematics, which in turn form two basic second order factors. Because of the lack of clear empirical or theoretical evidence supporting the above assumption, the data analysis was based on the theoretical model in Figure 1, which is, therefore, a data-oriented model.

Method

Variables: The model being tested contains both observed (measured) variables and latent constructs. The observed variables are specified as indicators for each of the latent constructs. The latent constructs were the result of a preliminary factor analysis of the questionnaire, in which four factors were identified as being the most appropriate in isolating four distinct scales relating to beliefs. The following is a brief description of the observed variables and latent constructs.

Content of mathematics (F1). This construct is measured with three items reflecting students' views about the content of mathematics, i.e., mathematics involves problems, calculations, constructions, diagrams, etc.

Nature of mathematics (F2). Three items are used as indicators of the latent construct students' beliefs about the nature of mathematics. The first item refers to students' beliefs about mathematics as a discipline that emphasizes fast and correct answers, the second to students' beliefs about mathematics as an abstract and strict discipline, and the third one to students' beliefs about mathematics as a subject that can be understood by all of them (Table 1).

Teaching of mathematics (F3). Modeled as a latent variable, students' beliefs about the role of teachers in the learning of mathematics are indicated by 2 items, each of which reflects teachers practices in the classroom. The first item refers to the teacher's role as the individual who helps students when they really need such help. The second one reflects the dominant role of the teacher as the individual who explains everything in the classroom or the individual who tells students what to do.

Learning of mathematics (F4). This latent construct is measured by 2 items reflecting the manner in which students view the learning of mathematics. The first item refers to students' beliefs about the way they learn mathematics, while the second one refers to students' beliefs about the effort needed in order successfully to meet the goals of mathematics.

Data Description: Data considered here are based on the Zimmermann-Pehkonen questionnaire (cited in Pehkonen, 1997) about students' beliefs towards the teaching of mathematics. Four random samples of students were selected to represent boys and girls in grades 6 and 9. Using listwise deletion of missing values in drawing the variables that were needed in the analysis, the final sample size was 660 students. Of these students 180 were males in grade 6 and 170 males in grade 9, while 190 were females in grade 6 and 101 in grade 9.

The analyses were conducted with covariance matrices, since the focus of the Confirmatory Factor Analysis (CFA) was to test the invariance of solutions across multiple groups, i.e., male and female students in grades 6 and 9. In the present study 10 belief indicators are hypothesized to represent four belief factors. We posited an a priori structure (see Figure 1) and tested the ability of a solution based on this structure to fit the data. Reflecting the relative importance of the different sets of parameters and the purpose of the study, we tested the following ordering of models which facilitates the comparison among different models and answers the questions of the study (Marsh, 1994).

1. Totally non-invariant model with no between-groups invariance constraints.
2. Factor loadings invariant across groups.
3. Factor loadings and regression correlations invariant.

Results

The Structure of Students' Beliefs: In answering the first question of the study, i.e., to find the structure of students' beliefs, we first tested the ability of the a priori model to fit the data simultaneously on the four groups with no invariance constraints (boys and girls in grades 6, and 9). The factor loadings of maximum likelihood estimation in our measurement model as well as the regression correlations are reported in Table 1. A satisfactory fit was achieved only after the introduction of first and second order factors (CFI=.910).

The findings indicate that students' beliefs towards mathematics can be represented in a sound way through both the first and the second order factors. The loadings on each factor (Table 1) were large and statistically significant, indicating that the first-order constructs are strong common factors that account for the observed variables. The results also show that the measures of the items under the model are highly reliable and replicate across the four independent samples.

The second order factors support the interpretation and use of beliefs as a multifaceted composite, and capture the structural organization of the belief system. The structural model shows that the two first-order factors (content and nature of mathematics) are intercorrelated through a second-order factor which can be thought of as an abstract representation of the overall students' views of the teaching and the learning of mathematics, and captures the shared variance across the two factors. The main advantage of the second order factor is that hypotheses can be tested about the hierarchical structure of students' beliefs towards mathematics. In the present study, the first-order factors "content of mathematics" and "nature of mathematics" can be represented by a second-order factor, which indicates students' "images" about mathematics. Students' beliefs about "what is mathematics" are significantly influenced by their beliefs about the teaching and the learning of mathematics. The hypothesized paths are significant for students in both grades (loadings: .705 for grade 6, and .718 for grade 9). In the same way, the first-order factors teaching and learning of mathematics measure the same hierarchical concept (teaching and learning), and each factor contributes to the estimation of the hierarchical factor in a specific way. The estimates of the latter second-order factor are all the same (.573) for 6th grade students, while in grade 9 the construct "teacher's role" (.448) contributes less than the constructs "student's role" and "practices" (.625).

Table 1:

Factor Loadings and Regression Correlations of the a Priori Model (without Constraints)

Factors	Items	Boys Grade 6	Girls Grade 6	Boys Grade 9	Girls Grade 9
F1: Content	Mathematics Teaching Involves:				
	Drawing figures (V1)	.504	.576	.681	.689
	Word problems (V4)	.567	.727	.493	.553
	Calculations (V8)	.490	.610	.552	.624
F2: Nature	Mathematics is:				
	Finding the right answer (V2)	.648	.950	.288	.646
	Strict discipline (V3)	.470	.467	.950	.352
	Understanding by all students (V5)	.292	.254	.188	.657
F3:Teacher's role	Teacher helps, when difficulties arise (V7)	.297	.113	.761	.207
	Teacher explains at every stage (V12)	.947	.949	.621	.563
F4: Learning	Mathematics require much effort (V9)	.380	.309	.600	.589
	Mathematics require practice as much as possible (V13)	.892	.828	.836	.604
F1, F5*		.766	.875	.893	.927
F2, F5		.213	.310	.218	.349
F3, F6*		.752	.717	.879	.931
F4, F6		.281	.341	.813	.904
F5, F7**		.795	.695	.864	.948
F6, F7**		.480	.229	.864	.919

* Second-order factor. F5=What is Mathematics, F6=TheTeaching and learning of mathematics

** Third-order factor. F7= Students' Belief System

The invariance of the a priori model across grade level and gender: The results provide good support for the hypothesized a priori model but do not address the issue of the invariance of the parameter estimates across grades and gender. Since literature in mathematics education suggests that there are gender and age differences among students in terms of their beliefs towards mathematics, we tested more specific hypotheses about the lack of invariance. To this end, we pursued two more specific tests. In the first, we constrained the factor loadings of the two groups to be equal, and in the second, we constrained the factor loadings and the regression correlations. Comparing the fit of the model with constrained factor loadings with the a-priori model shows that constraining the factor loadings to be equal in each of the four groups reduced the fit significantly. However, the fit of this model improved significantly, once some of the across grade level and gender constraints were released (CFI=.920; Table 2). This is indicated by the difference between the a priori model and the model with some factor constraints.

Table 2:

Goodness of Fit for Separate Solutions for Each Group with No Invariance and for Selected Invariance Constraints Imposed across All Groups

Model	df	x^2	CFI	df_d	x^2_d
No invariance constraints	203	269.53	0.910		
Factor Loadings Constraints	221	283.59	0.920	18	14.06
Factor Loadings and Correlations Constraints	234	292.65	0.930	13	9.06

Despite the differences in some indicators across grades and gender, there is good support for the invariance of regression correlations. For each grade level and gender, there is support that the structure of student's beliefs is invariant. Thus, introduction of invariance constraints on the regression correlations across all groups resulted in a change of x^2 of 9.06 (df = 13; Table 2) which suggests a small increment in fit (CFI=.930). Therefore, the hypothesized hierarchical structure of students' beliefs is invariant across groups, meaning that the belief system of students is created in their early school years.

Conclusions

The present study demonstrates that the proposed model supported the invariance of factor loadings, indicating that students responses were equally valid for boys and girls of different grade levels. The invariance of regression correlations indicated that relations among the different factors are the same for boys and girls of different grade levels and is relevant to the comparison of the gender-stereotypic and gender invariance models. In particular, there is no support for a gender specific pattern of correlations that varies with grade level, as it was hypothesized in many other studies. The invariance of factor loadings and regression correlations indicate that students' belief systems constitute a three level hierarchy and that the belief systems can be decomposed into two second-order factors: students' beliefs about the epistemological nature of mathematics (image of mathematics) and their beliefs about the teaching-learning process of mathematics. In turn, these second order-factors can be decomposed into four first-order factors: the content, the nature, the teacher's role and the learning of mathematics. This hierarchy also indicates that different factors influence the development of one's belief system towards the "good teaching of mathematics".

The invariance of the model also suggests that the belief system is formulated by the student experiences in primary school and it gradually develops in the same form in later years. Thus, students enter high school with a set of deep-seated beliefs, which are difficult to change. The structure invariance of students' belief systems is in agreement with the results of previous studies, indicating that early student experiences with mathematics have a significant effect on students' belief system.

In addition, the results of the present study indicate that change of the belief system is possible if we take into account the factors that mostly contribute to its creation and development. The present study shows that students' views about the nature of mathematics are a stronger predictor of students' belief systems than their views about the teaching and learning process, and thus change can be achieved by focusing on activities that help students to consider mathematics in a more meaningful way (see Table 1). This is in line with previous research indicating that content or curriculum changes have consequences on students' belief systems (Koupari, 1997).

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ABSTRACT SCHEMA VERSUS COMPUTATIONAL PROFICIENCY IN PERCENT PROBLEM SOLVING

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This paper explores the role of multiplicative comparison abstract schema in percent problem solving. It reports on Years 8, 9 and 10 students' knowledge of percent problem types, type of solution strategy, and use of diagrams. Non- and semi-proficient students displayed the expected inflexible formula approach to solution but proficient students used computational proficiency (a flexible mixture of number sense, estimation, conversions and trial and error) instead of the expected schema-based classification strategy.

When approached in terms of change from a mathematical perspective, percent problems can be considered an application of multiplicative comparison as follows.

INITIAL QUANTITY	*COMPARISON*	*GOAL QUANTITY*
40	------------------ × 3 ------------------>	120
40	------------------ × _ ------------------>	30
40	------------------ × 1.25 ------------------>	50
40	------- interest if rate is 25% ------->	10
40	------ resale price if 25% profit ---->	50

Multiplicative comparison situations can have an unknown in three positions, thus leading to three problem types and corresponding solutions: (a) when the goal quantity is unknown, the solution requires multiplication; (b) when the initial quantity is unknown, the solution requires division; and (c) when the comparison is unknown, the solution requires division. Thus, multiplicative comparison and its relation to the three problem types represents an abstract schema (Ohlsson, 1993) and appears to make the solution of one-step percent problems straightforward (i.e., identify the initial and goal quantities and the comparison, determine which is unknown, and calculate appropriately). Consider, for example, the problem, *Jack bought a house. He paid a 35% deposit of $210 000. What was the price of the house?*. When this is rethought in terms of multiplicative comparison, the unknown is seen to be the initial quantity (? ---- × 0.35 ---> $210 000) and, therefore, the solution is found by division

The three abstract schema problem types have been categorised in the literature as Type A, B and C problems (Ashlock, Johnson, Wilson, & Jones, 1983).

Type A	Percentage unknown - finding a part or percent of a number, for example, what is 25% of $60 (i.e., *goal quantity unknown*);
Type B	Percent unknown - finding a part or percent one number is of another, for example, what % of $60 is $15 (*comparison unknown*);
Type C	100% unknown - finding a number when a certain part or percent of that number is known, for example, 25% of the cost is $15, what is the total cost (*initial quantity unknown*).

However, do students invoke some form of multiplicative abstract schema to solve the three problem types?

Percent problem solving

A knowledge of percent is vital in many facets of the real world. For example, percent discounts, profits, losses, savings, and increases are an integral part of our society. Because of this social necessity, percent is an important component of the mathematics curriculum and has been the focus of curriculum planning, observations and research studies. Four main strategies for solving percent problems have been identified: *cases*, *proportion*, *unitary*, and *formula* (e.g., Dole, Cooper, Baturo, & Conoplia, 1997; Parker & Leinhardt, 1995).

> *Cases* - students classify the problem and apply a different procedure for each problem type (multiply the number by the percent as a decimal for Type A, divide the numbers and translate the decimal answer to a percent for Type B, and divide the number by the percent as a decimal for Type C).
>
> *Proportion* - students classify the problem in terms of the unknown, consider percent as a common fraction with a denominator of 100, equate this to a fraction made up from the two other possible numbers (i.e. a/100 = c/d), and find the solution by algebraic manipulation or cross-multuply method (the "rule of three").
>
> *Unitary* - students calculate 1% of the "known" and calculate the required percent by simple arithmetic computations (e.g., 11% of 200 is thought of as the product of 1% of 200 and 11).
>
> *Formula* - students substitute into a formula (e.g., P = BR, where P is percentage, B is base number and R is percent) and find the unknown by algebraic manipulation.

The concept of percent can be modelled with 10x10 grids (a large square divided into 10 rows of 10 small squares) or number lines (from 0 to 100), and these models can help students visualise the computational procedures of percent calculations (e.g., Bennett & Nelson, 1994). Mnemonic strategies, which emphasise the *key words* "of" (meaning multiply) and "is" (meaning divide), have also been suggested to help students interpret percent problems and to order percent calculations (e.g., McGivney & Nitschke, 1988).

The literature (e.g., Kouba et al., 1988) has confirmed that students perform poorly on percent, problems particularly Types B and C problems (unknown is the multiplier or the first quantity). Percent is seen as a confusing topic in the mathematics curriculum for both students and teachers (Parker & Leinhardt, 1995). Furthermore, the results of comparative teaching studies do not conclusively suggest that one instructional method is superior to another (Parker & Leinhardt, 1995), but indicate that instruction does effect performance and that prior arithmetical knowledge assists solution. For example, Lembke and Reys (1994) found that Years 5 and 7 students (no instruction in percent) used a variety of intuitive strategies, Years 9 and 11 students used formulae, and all students used common benchmarks (e.g., 50% is half, 25% is half of a half) to aid calculation and check reasonableness of calculations.

Problem-solving success and solution strategy

Understanding percent requires appropriate mental models to accommodate its "multiple and often embedded meanings and its relational character" as well as the procedures solving percent problems (Parker & Leinhardt, 1995, p. 47). Students can have knowledge that is *syntactic*, the correct performance of mathematical procedures; or *semantic*, the understanding of the meaning of those procedures (Resnick, 1982). Their knowledge can be *intuitive*, "everyday" real world application knowledge normally acquired before instruction; *concrete*, associated with representation by appropriate concrete materials during instruction; *computational*, knowledge of the algorithmic procedures; or *principled/conceptual*, knowledge of the principles that constrain/justify algorithmic procedures normally taking place after instruction (Leinhardt, 1986). They can have knowledge that is *operational*, dynamic sequential and detailed; or *structural*, abstract, static, instantaneous and integrative (Sfard, 1991).

However, to solve problems, students also need to access this knowledge. Prawat (1989) argued that access to knowledge is determined by the learner's organisation and awareness of three factors: *knowledge*, concepts, principles, rules, facts and procedures; *strategic and metastrategic thinking*, general problem solving heuristics and metacognitive processes (e.g., planning, monitoring, checking, revising); and *disposition*, habits of mind. In particular, as Garofalo and Lester (1985, p. 167) argued, accessing mathematical knowledge is influenced by three metacognitive categories of *person knowledge*, "one's assessment of one's own capabilities and limitations with respect to mathematics in general, and also with particular topics or tasks" including such affective variables as motivation, anxiety and perseverance; *task knowledge*, one's beliefs about the nature of the mathematical tasks; and *strategy knowledge*, awareness of strategies for guiding problem solving. Thus, adequate percent knowledge consists of the meanings of percent in its many dimensions, principles which legitimise percent calculations, and metacognition to enhance access.

Therefore, it seems reasonable that students who can successfully solve all percent problem types have semantic, principled-conceptual and structural percent knowledge and strategic and metastrategic thinking, including self belief and strategy knowledge. It also seems reasonable to suppose that students with this knowledge would have some form of abstract schema with respect to percent problem types and multiplicative comparison, be able to access this knowledge when solving one-step percent problems, and use solution strategies that efficiently translate a schema-based understanding of percent to solutions, most likely, the *cases* strategy. On the other hand, it seems reasonable that less successful students would be unable to access knowledge useful to percent problem solving, have syntactic, procedural and operational percent knowledge and lack strategic and metastrategic thinking, and use rote procedures inflexibly in attempting to solve percent problems, particularly the *formula* strategy.

This paper reports on a study (Dole et al., 1997) that attempted to see if the expectation that success and abstract schema are related holds for junior secondary students and percent.

Study

Ninety students from three classes (Years 8, 9 and 10) from a Brisbane secondary boys school were given examples (designed to be within the experience of the students) of the three types of one-step percent problems to solve and, from their responses, were categorised as: *proficient*, able to solve all three types of percent problems; *semi-proficient*, able to solve Type A problems but not able to solve Types B and C problems; and *non-proficient*, not able to solve any type of problem. A sample of 18 students, evenly representing all proficiencies and year levels, was given a semistructured clinical interview. The tasks focused on students' understanding of percent problems and the strategies the students used in solving these problems. The first task explored students' schema of percent (i.e., their knowledge of the three percent types) by asking how many different types of percent problems there were; the second identified strategies used by the students in solving percent problems by asking them to show how they solved the three problem types; the third explored students' visualisation of the percent problems by checking on their use of diagrams, and by asking them to solve problems with diagrams if they had not spontaneously done so.

The students were removed from their class and interviewed in a separate room. The interviews lasted 30 minutes and were videotaped. The students had attempted the problems before the interview and the interview focused on recalling the methods they had used in solution. If knowledge was detected that had not been used in problems, the students were questioned as to why it was not used.

Results

The eighteen students are denoted as follows (the first number refers to their Year level). It should be noted that only 6 proficient students were identified.

Proficient students	8P1, 8P2, 9P1, 9P2, 10P1, 10P2
Semi-proficient students	8SP1, 8SP2, 9SP1, 9SP2, 10SP1, 10SP2
Non-proficient students	8NP1, 8NP2, 9NP1, 9NP2, 10NP1, 10NP2

Students' responses on number of percent problem types. Four of the six proficient student (8P2, 9P2, 10P1, 10P2) and two semi-proficient students (9SP2 and 10SP2) stated that there were three problem types and identified them as the three multiplicative comparison abstract schema problem types (i.e., Types A, B and C). Two students (8SP2, 8NP2) identified problem types A and B. (8SP2 also identified two extra problem types which he described as "profit problems" and "loss problems".) The remaining 10 students (8P1, 9P1, 8SP1, 9SP1, 10SP1, 8NP1, 9NP1, 9NP2, 10NP1, 10NP2) did not identify any of the three problem types; rather, they used syntactic categorisations based on context (e.g., "questions on maths tests", "percent in the real world", "percentages used to sell things", "percentages used for exporting", and "those ones which you divide and multiply"). The non-proficient students were particularly creative in their categorisation of problem types.

Students' solution strategies. All the percent solution strategies (*cases, proportion, unitary* and *formula*) were in evidence with the most widely used being *formula*. The proficient students used a variety of percent strategies: 8P2 and 10P1 used the *cases* strategy, 8P1 and 9P1 used the *unitary* strategy, and 9P1, 9P2 and 10P2 used the *formula* strategy. The semi-proficient students used less variety: 8SP1 and 8SP2 used the *proportion* strategy, and 9SP1, 9SP2, 10SP1 and 10SP2 used the *formula* strategy. The non-proficient students showed no variety; all used the *formula* strategy. (In fact, if the non-proficient students could not determine a formula, they did not attempt the problem.)

Other strategies were used in tandem with the percent strategies, namely, *trial-and-error* and *key words*. Two proficient students (9P1, 9P2) and three semi-proficient students (9SP1, 10SP1, 10SP2) used the *trial-and-error* strategy when they could not remember or work out the formula. Four non-proficient students (9NP1, 9NP2, 10NP1, 10NP2) used the *key words* strategy where "of" indicated multiply and "is" indicated divide (McGivney & Nitschke, 1988) to assist them to determine what the formula might be.

As well, number sense was widely used in the solution of the problems. Proficient students showed strong skills in mental computation and operation relationships (8P1, 9P1, 9P2, 10P1), conversions between percent, common fractions and decimal fractions (8P1, 10P1, 10P2), and benchmarking, approximation and estimation (8P1, 8P2, 9P1, 9P2, 10P1). For example, in the Type B problem, "186 is what percent of 240", 9P1 argued that the solution was larger than 50% because 50% would be 120 out of 240; in the Type A problem, "28% of 150", 10P1 argued that the answer should be close to 50 because 28% is approximately 1/3; in the Type C problem, "51 is 85% of what number", 8P1 said that the answer had to be a little larger than 51 because of the relationship of 85% to the whole.

Semi-proficient students also used benchmarking, approximation and estimation to assist their problem solving. However, unlike the proficient students who used these skills formatively (on the way to a solution, and often with the *trial-and-error* strategy), semi-proficient students tended to use the skills summatively (for checking answers). Furthermore, they were not as skilled as the proficient students in mental computation were. Two semi-proficient students (8SP2, 9SP1) were skilful with conversions.

Non-proficient students did not generally reveal flexible thinking; rather, they tended to follow routinised patterns of activity (e.g., converting the percent to a decimal) even if that activity was not helpful in solving the problem. For some non-proficient students, the activity was so automated that they did this before they read the problem through.

With respect to structural analysis, four proficient students only (8P2, 9P1, 9P2, 10P1) indicated an ability to consider problems from multiplicative comparison abstract schema, to classify problems by type, and to use this classification in their solution procedure. These proficient students tended to use structural analysis in tandem with estimation and the trial-and-error strategy. For example, when given two numbers and asked to find a percent, 9P2 divided the smaller number by the larger, multiplied by 100, looked at the answer, and then reversed what he had done when he thought the answer was unreasonable.

Students' responses with respect to use of diagrams. No students spontaneously drew diagrams as part of their solution procedures. When asked to use this strategy to help solve problems, all the proficient students, one semi-proficient student (9SP2) and five non-proficient students (8NP1, 8NP2, 9NP1, 9NP2, 10NP2) were able to draw diagrams that reflected the problem. The remainder could not think of an appropriate diagram to draw and resisted the interviewer's request. Of the students who drew diagrams, four (8P2, 9P1, 8NP2, 10NP2) drew number lines, three (8P1, 10P2, 9NP2) drew pie charts, three (10P1, 9SP2, 9NP1) drew 10x10 grids, one (8NP1) drew rough rectangles, and one (9P2) drew diagrams of rivers (!) and used the analogy of people crossing these rivers to assist him to solve the problems.

Conclusions

Proficient students generally knew that there were three types of problems. They could also represent these problems with a variety of effective diagrams when asked to do so. With regard to their solution strategies, they found it frustrating to discuss their procedures for solution and their preferred response was "I just do it!". They were much more flexible than the less proficient students in that they did not rely solely on the *formula* strategy but tended to use a variety of strategies and procedures. Furthermore, if a strategy was unavailable or ineffective, they tried another strategy. In their solutions, they constantly estimated and manipulated numbers until the answers made sense, converted readily between percents, common fractions and decimal fractions, and had a good understanding of the relative size of numbers in terms of relationships in the problem (in this, they tended to have the multiple meanings of percent as described by Parker & Leinhardt, 1995). They appeared to have good mental calculation skills and to understand the effect of operations (e.g., they reversed operations). Importantly, they also appeared to be able to analyse problems in terms of their meanings, predict the required operation and gauge the size of the answer relative to the numbers they had been given (e.g., when given the Type C problem, "51 is 85% of what", they could see that 51 was approximately 3/4 of the answer).

Except for one student (9SP2), semi-proficient students had no idea of the number of percent problem types and could not represent percent situations with diagrams. With respect to problem solutions, they were reliant on the formula strategy although they were happy to use trial-and-error if they forgot the formula. Like the proficient students, they also used benchmarking, approximation and computational estimation but, unlike the proficient students, usually as a checking mechanism rather than as an aid for analysis and *prediction. They were able to* realise when an answer did not make sense, but were unable to construct alternative strategies to correct their mistakes or overcome difficulties.

Non-proficient students thought there were many types of percent problems, usually seeing surface features as constituting difference (e.g., percent to sell things and percent to import were different problems). Surprisingly, they were able to draw (when asked) appropriate diagrams to represent problems but were unable to use the diagrams in the solution procedure. They routinely attempted to solve problems by the only strategies available to them, namely, *formula* and *key words*, regardless of their appropriateness. Thus, their approach to percent problem solving was syntactic in nature (Resnick, 1982) and limited. (If they could not apply a formula, they could not attempt the problem.) They had poor estimation and computational skills, and could not tell if an answer was sensible.

Expectations and implications

This study explored the expectation that percent problem solving success and multiplicative comparison abstract schema are related. That is, proficient students would use a schema approach to percent problems while less proficient would use inflexible rote procedures. The expectation held for the less proficient students but not for the proficient students.

The proficient students' problem solving behaviour did not strongly reflect a schematic understanding although they did show some indication of structural analysis and they did know the three problem types. They used strategies and metastrategies and were confident in their solutions (as expected from the findings of Garofalo & Lester, 1985). However, instead of a schema-based interpretation of problems leading to a classification approach to solution (e.g., the *cases* strategy), they tended to use what could be called a "first principles" approach to solution in which the *trial-and-error* strategy, and computational proficiency (number sense skills such as mental computation, estimation and benchmarking, and conversions) are used with a flexible mixture of percent strategies (*cases*, *unitary*, *formula*).

The semi- and non-proficient students' behaviour did reflect expectation. Both groups of students were inflexible and formula oriented. The semi-proficient students used some estimation and trial-and-error while the non-proficient students focused on key words and conversions, and discontinued solution attempts if they could not determine an appropriate formula.

This leads to two implications and a dilemma. The first implication is related to improving semi- and non-proficient students' performance to that of proficient students'. Thus, number sense (including mental computation), benchmarking, approximation, estimation, and conversions between percent, decimal fractions and common fractions should be the focus of explicit instruction, along with the *trial-and-error* strategy. Computational proficiency should be seen as a necessary prerequisite for percent problem-solving proficiency.

The second implication stems from the students' general inability to solve Type B and C problems (only 6 of the 90 students tested could solve these problem types). It is likely that students in lower grades have similar difficulties for these problem types when related to whole-number and fraction multiplicative comparison situations (e.g., John has 5 times the money that Fred has, John has $60, how much has Fred?) Thus, Types B and C problems should be part of instruction for whole number and fraction multiplication. Furthermore, connections should be drawn between whole number and fraction multiplication problems and percent problems for all problem types.

The dilemma is whether instruction should be taken beyond these two implications. Should we be satisfied with students able to solve percent problems from the "first principles" of computational proficiency, or should we develop a curriculum that attempts to construct an explicit multiplicative comparison abstract schema for all students? Is such an abstract schema a worthwhile end for mathematics instruction, particularly when this study has shown that students can be successful without it?

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IMPLICIT COGNITIVE WORK IN PUTTING WORD PROBLEMS INTO EQUATION FORM

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In this research, we focus our analysis in the putting into equation process in which a single-unknown equation can be written. We constructed a classification of word problems that appear in school text books for 8th-grade classes. The solving of several problems is analysed to demonstrate difficulties encountered, which are often inherent to each category of problems. Implicit cognitive work involved in the putting into equation process is modelled.

Several authors have studied the solving of word problems among them: Bloedy-Vinner H.(1996), MacGregor M. & Stacey K.(1996), Nesher P. & Hershkovitz S. (1994), Schoenfeld A. H. (1985). The importance of the concept of function, in solving word problems, has been put forward by several authors, among them: Rojano T. and Sutherland R.(1993); Nathan M.J, & Kintch W. and Young E.(1992); Yerushalmy (1997)... In agreement with these authors, we investigate how the concept of mathematical functions, which is often implicit, plays a role in this process. Though a complete analysis of these studies is impossible within the framework of this research.

Statements of problems proposed to students can be considered as a specific representation of a fictitious or a real situation; the term *reference* is used to indicate each of the different types of situations used. Linguistic difficulties, that students may encounter, will not be analysed because, in general, the comprehension of problem statements that we use do not pose major difficulties to our students.

Students' written equations are mathematical representations of these situations in which unknown numbers must be calculated. We collect these equations which are written traces, though from the reading of the problem statement and the writing of the equations, students go through a considerable amount of *implicit cognitive work*; the nature of which occupies the space spread out between what is not expressed and what is not expressible or even no-conscious.

In this research, *we focus our analysis in the putting into equation process in which a single-unknown equation can be written*. We constructed a classification of word problems that appear in school text books for 8th-grade classes. The solving of several problems is analysed to demonstrate difficulties encountered, which are often inherent to each category of problems. *Implicit cognitive work involved in the putting into equation process is modelled.*

Theoretical framework:

Our theoretical framework is based on the "Conceptual Field Theory" Vergnaud, G. (1990). Cognitive behaviour is modelled in terms of schemes ("schème"): the organisation of behaviour (action) for a certain class of situations. The functioning of the scheme is based, essentially, on operational invariants: knowledge, often implicit, that allows the search of relevant information, to make inferences and anticipations. Starting from the analysis and the classification of errors in the putting into equation process, implicit operational invariants have been identified in Cortés (1995). Operational invariants is mathematical knowledge, the tools used in the construction of a mathematical representation.

Operational invariant in the putting into equation process:

Problem statements that appear in school text books use references (pebbles, lengths, passengers...) whose additive and multiplicative properties students, in general, know. Comparisons of measures are indicated, in natural language, by conventional sentences as: "the length has 220m more than the width" or "3/10 of yellow tulips", "four times the price..." that are, *in general*, enough well understood by our students. After reading the word problem, students are faced with constructing or identifying useful correspondences between spaces of measures, expressed in natural language. These correspondences are particular cases of mathematical functions expressed in natural language. The student uses an operational invariant: **the implicit and pragmatic concept of modern mathematical function** (to a measure corresponds only one measure inside an other space). It is implicit operational knowledge because students never seen this definition explicitly.

For example in the following problem: ***I - 1*** *- A merchant of fruit receives* **m** *Kilograms of peaches. In the course of the sale he must throw out 3 kg of damaged peaches. The merchant sells the remaining peaches at the price of 15F by Kg and the cash results 450F. What is the number of Kg of received peaches ?*

This problem states correspondences between the weight space (Kg) and the price space; an explicit one is: 1Kg--> 15F; and another is easily inferred: (m-3) Kg--> 450F.

With the identification of these correspondences a numerical function can be implicitly constructed, from a well known property of prices: prices (P) are function of the unit price (p) and the number of objects (n), thus the equivalence p ni = Pi is a particular case of the numerical function P= pn. Of course, students construct this function mentally, according to theirs pragmatic representations using an operational invariant: **the concept, often implicit, of numerical function** (it is implicit knowledge because student did not study linear functions before). By implicit substitution of unknowns with given numbers or using it in constructing a particular case of the numerical function; students can write an equivalence (f.e. 15(m-3)=450). The writing of an equation (equivalence of measures) necessitates the use of **the concept of equivalence** (another operational invariant). The introduction of the "equal" sign implies a rupture with natural language and the passage to the algebraic representation is guided by another operational invariant, a principle: **"the respect of the homogeneity of terms"**(homogeneity of units and symbol signification).

The experimental work: our students, belonging to three 8th-grade classes, are able to solve certain word problems using algebra, but their method is limited to the writing of a single unknown equation. The three classes have been taught of the same manner: *the unknowns that appear in word problems are expressed as functions of the unknown that one is to calculate by means of natural language* , for example: "x is the weight of Jacques; x+220 is the weight of John". Most of the students only write the final single-unknown equation. The three 8th-grade classes were tested: 20 word problems were given to students (5 different tests comprised of 4 problems each); each problem was solved by approximately 18 students (6 students in each class). Two classes presented analogous success rates, while the third, far more weak, had an inferior success rate for each problem.

CLASSIFICATION OF PROBLEMS

In the present research, the writing of a single-unknown equation will be the outcome of the putting into equation process, because it is the only mathematical object that our students know. This written equation is a mathematical representation of the word problem and it provides a means to calculate the numerical value of the unknown. By taking into account that the outcome of the mathematical representation of a problem is always a single unknown equation we have classified word problems that appear in school text books for 8th-grade classes in three categories.

It is possible to construct a single-unknown equation by:

I - substituting unknowns with given numbers and units into a given formula or into a constructed function.

II - substituting unknowns with linear functions into a two (or more) unknown equation.

III - equating two linear functions.

These substitutions are, in general, implicit cognitive processes.

I - Mathematical representation of problems in which it is possible to construct a single-unknown equation by substituting unknowns with given numbers into a given formula or into a constructed function.

The previous problem belong to this category. In solving this problem 23% of students wrote the following equation: 15 (m-3)= 450 in which the quantity (m-3) Kg is processed implicitly as a "number of objects" and consequently, the units of the term:
15(F/Kg)(m-3) Kg are considered as Francs. Likewise, in the next equation m=3+(450/15), the quotient (450/15) is considered as a number (without units) having the meaning of "number of Kg". Sixty-five percent of students responded with this type of solution, being close to the arithmetic calculation. They respect the homogeneity principle.

However, in the following problem the processing of units has to be made explicitly:

I - 2 - A train passes ahead of an immobile observer in 6 seconds and crosses a tunnel of 2 Km in 60 seconds. What is in Km/ Hour the average speed of the train and what is its length?

Fifty six percent of the students calculated the speed; most of them, by establishing correspondences between time and distance spaces 60 s--> 2 Km, 1 min.---> 2 Km; 60 min.--> 120 Km; thus, the speed is considered as the distance (120 Km) covered in one hour. Only two students wrote the speed as a quotient of measures V=2Km/1min then transform minutes into hours.

The calculation of the length of the train then becomes very difficult, only 19% of students manage to calculate it. They establish correspondences between time and distance spaces 60 s---> 2 Km; 6 s--> (6* 2/60) = 0.2 Km. Only one student used the formula d=vt by replacing data and by respecting the homogeneity constraint. Most errors consist in using data, dissociating numbers from their units which are accounted for at the end of numerical calculations or ignored: the homogeneity constraint is not respected.

The resolution of the following problems implies the construction of functions and equations comprising several additive terms: states and transformations *which can be* *negative*. We are particularly interested in the respect of coherence between the meaning (and the sign) given to the unknown and that of the obtained numerical results: *the respect of the homogeneity of symbol signification*. **I - 3** - *Robert has played two marble games. He had 15 marbles at the beginning. At the end of the game he had 27 marbles more than what he had had at the beginning. He no longer remembered what happened during the first game, during the second game he has earned 35 marbles.* In this problem, on the one hand, the final state is an additive function of the initial state and of the successive transformations (Ef = Ei+t1+t2) and its numerical value can be calculated: Ef=Ei+T=15+27. On the other hand T is a function of the other transformations: T= t1 + t2 (students did not use this relation). The nature of the unknown transformation (loss) is not evident. Students mentally construct this function in their representations. Thus, 33% of students wrote 15+x+35=15+27 ... **x=-8** and end the problem by the sentence: "he has lost 8 marbles"; the negative sign of the result has the meaning of a loss; the coherence of the process is kept. This coherence is also kept by a student that introduces the meaning of loss in the definition of the unknown as well as in the equation that he writes x= (35 + 15)-(27 + 15)... x= 8. Several students elude the problem of the sign of the unknown (as well as the reiterated utilisation of the data Ei= 15 into the equation) by calculating an intermediate state: "x is the number of marbles at the end of the first game"; x+35=15+27... x= 7 and then he writes "therefore, he has lost 8 marbles".

There are errors of coherence between the signification given to the unknown and the sign of the result, for example: "x is the number of lost marbles", this signification is introduced in the equation x+15+27= 15+35; x=-42+50; **x=-8** where the wrong sign of the result can be considered as a manner (not conscious) of again finding the meaning of loss. Several students wrote *wrong equivalencies*, they "refuse" to use the data (Ei= 15) twice, leading to the following solution: x=15+27-35 ... x=7 and "he has lost 7 marbles"; an intermediate state is confused with the unknown transformation. In the equation 15+35+x=27 the resulting transformation is confused with the final state.

Problems	**I - 1**	**I - 2**	**I - 3**
Success rate	88%	19%	60%

Observations: In word problems involving multiplication of measures, if the independent variable of the proportion can be processed as a number (without units), the quotient of measures disappears and the multiplication of measures becomes the multiplication of a measure by a number. The mental construction of a numerical function and the writing of an equation as a particular case is then facilitated (I-1). On the other hand, the difficulty to explicitly process multiplication and quotients of units often induces students to work only with correspondences between two spaces of measures (I-2). They use implicit knowledge, an operational invariant always present in mathematical representation, the pragmatic mathematical function concept expressed in terms of correspondences. This type of problems is studied in Nathan M.J. & Kintch W., Young E.

In word problems involving the addition of measures, students construct an implicit representation of a function of several variables that serves as basis for written equations. In cases where the nature of the unknown transformation (gain, loss..) can be easily identified, the rate of success considerably increases. They use implicit knowledge, an operational invariant always present in algebraic representations: the pragmatic numerical function concept (I-3).

II - Mathematical representations of problems in which it is possible to construct a single-unknown equation by substituting unknowns with linear functions into a two (or more) unknown equation. This type of problem is very frequent in school text books.

II - 1 *- Here are two piles of pebbles. The second pile has 22 pebbles more than the first and there are 528 pebbles in all. How many pebbles are in the first pile?*

In this problem it is necessary to process two unknowns, the number of pebbles in the first pile (N1) and that of the second (N2). The knowledge of additive properties of numbers of objects provides a means to ***implicitly construct*** the function "total number ($Nt=N1+N2$)" as well as the two unknown equations $528 = N1+N2$. The number of pebbles in the second pile, as a function of the unknown that one wants to calculate ($N2= N1+22$), is also constructed ***in students' mental representations*** and substituted. All students succeed in solving this problem, most of them write: $x+x+22 = 528$. A student writes a function by means of the notation that he/she knows: "the number of pebbles of the second pile is $(x+22)$".

II - 2 *- Two twins, John and Jacques, weigh 5.28 Kg. together. John weighs 220 gr. more than Jacques. How much does each of twins weighs?*

This problem is analogous to the precedent. There are two unknown weights P1 and P2. The functions: "total weight" ($P=P1+P2$) and the second weight related to the first $P1=P2+220$, are constructed ***in students' representations*** using the knowledge the students have about additive properties of weights. In this problem, respecting the homogeneity of terms implies the conversion of data to the same unit. Thus, students that succeed are more explicit; they write, for example: "x is Jacques' weight; x+220 is John's weight; 5.28Kg= 5280 g; $x+x+220= 5280$". There are errors in the conversion of units; for example: 220 g becomes 2.2 Kg. The next erroneous equivalence $(5.28-x) = (5.28-x)+220$ (which also does not respect the homogeneity of terms) implies the mental transformation of $5.28= P1 + P2$ into $5.28-P1= P2$ (we used our notations) which is then substituted into $P1= P2 + 220$ in a wrong manner. The errors in this long implicit process makes it obvious that memory has considerable limitations.

II - 3 *- A gardener wants to plant a bed of tulips in which there would be 3/10 yellow tulips, 2/10 red tulips and 30 black tulips. How many tulips did the gardener buy?*

The resolution of this problem implies the processing of three unknowns and the construction of three functions; it is not succeeded as well the others. The total number of tulips is a function of the other numbers of flowers ($x= Ny + Nr + 30$). Unknown numbers of red and yellow tulips are a "part" of the total number Ny---> (3/10) x; Nr---> (2/10) x. By means of implicit substitution one constructs the calculable single-unknown equation. Only 50% of the students succeeded using algebra: $(2/10)\,x + (3/10)\,x + 30= x$. Several students (25%) proposed an arithmetical solution, for example: $30=(10/10)-(3/10)-(2/10)$; $30= (5/10)$, $6=(1/10)$, $60= (10/10)$, ***where the equal sign is used to note correspondences.*** Twenty-five percent of the students process fractions as numbers of tulips:

(3/10)+(2/10)+ 30= x; x= 30.5; the meaning of fractions "part of a whole" is not taken into account. This problem is similar to those analysed in MacGregor, M. and Stacey, K. (1996) and in this research "reversal errors" were not important; also, in our research "reversal errors" do not appear.

II- 4 *- One pays the sum of 1750F with 24 bills of 50 and 100 Francs. How many bills are there of each kind?* This problem is too difficult for most of our students. Two students wrote: 1750= 100x + (24-x) 50; the expression (24-x) is the written trace of a long implicit process (using author's notations): x+y= 24 transformed into y= 24-x, then substituted as the addition of two sums of money S1+S2=1750 being S1= 50x et S2= 100y. Most wrong equivalencies do not take into account the constraint concerning the number of bills (*a function is not constructed*), for example: "x bills of 50F and 2x bills of 100F", 50x+200x= 1750;...x= 7F. There is only one "analgebraic" error (Bloedy-Vinner (1996)) made by the unique student that uses two unknowns explicitly: 50x+100y= 1750xy...impasse. One can formulate the hypothesis that this type of error appears when students undertake a conceptual jump, for example, when they begin to process explicitly two unknowns.

A student proposes a solution based on numerical calculation of particular cases of the mathematical function: *24 bills----> corresponding sum of money*. He calculates x=y=12 and 600+1200=1800... x= 13 y= 11 and 650+1100=1750. This student used implicit knowledge: the pragmatic mathematical function concept *expressed as correspondences* (between number of bills and sums of money) using natural language and arithmetic calculations.

Problem	**II - 1**	**II - 2**	**II - 3**	**II - 4**
Success rate	100%	81%	75%	19%

Observations: the problem statements concern measure (pebbles, passengers, lengths...) which additive and multiplicative properties, *in general*, students are familiar with. The respect of the homogeneity constraint increases the difficulty (II-2). Furthermore, when the implicit construction of functions and substitution becomes too difficult or too long; many students (some because of memory saturation) lose control of the process (II-5).

III - Mathematical representation of problems in which it is possible to construct a single-unknown equation by equating two linear functions.

This type of problem is less frequent in school text books. These problems involve difficulties, notably processing several unknowns. Solving them implies, at least, the mental construction of two functions (for example y=ax+b; z=cx+d) and the use of a condition (for example y= z) to construct an equation (ax+b= cx+d). This type of problems is used in Yerushalmy M. (1997).

III - 1 *- A father is 30 years old and his son is 4 years old. In how many years will the father's age be the double of his son's age?* The additions of an unknown time lapse (x) to initial states provides a means to construct the implicit functions: "father's age---> 30+x", "son's age----> 4+x" and "twice the son's age---> 2 (4 + x). The problem also states a condition, "the father's age---> twice the son's age", and this condition provides a means to construct the problem equation, for example $30+x= 2(x+4)$. In this case, 56% of students wrote the correct equation. The implicit construction of three functions and constraint respect seems too long for some students. Thus, two students wrote only multiplicative relationships $30.x=2.x.4$ (in which the resolution does not respect the priority of multiplication: $30x-2x= 2x.4-2x$; $28x= 4$; $x= 7$). Other students are not able to take into account the condition between final states; they write, for example, $30+x=4+x$ (impasse). Twenty percent of the students do not solve the problem.

Students solved the following problem better. ***III - 2*** - *To rent a car two possible contracts are available: A) Pay 3F for each Km. driven. B) Pay 200F for the rental and 2F for each Km. driven. For what distance will the two contracts be equivalent?*

The first given correspondence (contract A: 1 Km---> 3F) allows, for most of the students, to mentally construct a proportion (xKm ---> 3x). This proportion is globally perceived as having the meaning of a sum of money; the analysis of units in the multiplication of measures: 3(F/Km) * x Km is out of reach for our students: the xKm are considered as a number. In contract B the proportion is easily constructed (1 Km--> 2F, xKm--> 2xF) as well as the addition of the two sums of money suggested by the problem statement (students know additive properties for sums of money). Thus 68% of the students wrote the equivalence $3x= 2x+ 200$ (x being designated as number of kilometres or as a distance in Km) and then calculate the solution. For all types of errors, students construct the two functions. For example, a student considers providing two different distances corresponding to the same sum of paid money as a solution: A= 134 Km, 3*134= 402F and B=101 Km, 200+2*101=402F. An other writes: $x= 3x$; $x= 200+2x$ and "I do not know how to do it"; *the writing of numerical functions need two variables and he uses only one.*

Problem	**III - 1**	**III - 2**
Success rate	56%	68%

Observations: A good number of students do not feel capable at solving this type of problem, stating that "there are several unknowns...". The implicit construction of functions is easily made for certain problems (III-2). However, problems become more difficult if the number of functions increases (III-1) or if the functions are more complex (for example, $y=x+2x+4x+8x$ and $y=x+(x+22)+(x+44)+(x+66)$).

CONCLUSION

Most of our students are able to put into equation form many of these problems, errors of the type "syntactic translation" (MacGregor and Stacey (1993)), frequent at the beginning of the learning process, do not appear here. The reversal error (for example: $x=y+20$ instead of $y=x+20$) and "analgebraic" errors (Bloedy-Vinner H.(1996)) are frequent when both unknowns (x and y) have the same status and are used explicitly in the construction of systems of equations Cortés (1995). Consequently, "reversal errors" does not appear in this research and neither does it in Macgregor M. and Stacey K; (1996) research. Written equations must respect "the homogeneity of terms". We observe that the respect of this principle is a conceptual construction that some of our students do not have.

We notice that when the problem is too easy (I-1), or rather difficult (II-3), many students use arithmetic. For a very difficult problem (II-4) half of them do not solve it, and we make the hypothesis that, often, errors are due to the saturation of memory. The existence of superfluous data provokes a great number of failures; it can be hypothesised that a good number of students use rules of action such as: "Use all the datas". Students often organise their behaviour by taking into account school phenomena that do not concern either mathematics or properties of references.

The analysis of student's cognitive behaviours shows that the function concept plays a very important role and we observe, at least in France, that the study of functions is made after and separately from the study of equations and the resolution of word problems. In agreement with Yerushalmy (1997): "the function is the appropriate fundamental object of secondary school mathematics, and focusing on the concept of function allows the organisation of algebra curriculum around major ideas rather than technical manipulations".

This research has a dual purpose, on one hand, it provides a contribution to the theory of implicit cognitive processes, on the other hand, it provides a contribution for the construction of teacher training courses.

The classification of word problems and the model of the student's implicit processes that we propose can become a daily work-tool for teachers. The teachers' mediation role frequently includes the analysis and the correction of errors, it is therefore capital for them to precisely infer what is missing or what is incorrect in the student's mental representations in solving problems.

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"THREE SIDES EQUAL MEANS IT IS NOT ISOSCELES"

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Class inclusion, identified relationships between figures based on their properties, is an important characteristic of Level 3 thinking in the van Hiele theory. While researchers have highlighted the difficulties associated with understanding this idea and, the importance of class inclusion as a prerequisite for deductive reasoning, there needs to be further research directed at how class inclusion concepts evolve. This study, involving indepth interviews with 24 secondary students, addresses this issue by considering their attempts at linking and grouping seven different triangle types. The purpose of this paper is to report the findings of an initial exploration into the quality of the students' responses to the task in an attempt to provide light on the difficulties students face with class inclusion. The findings reveal important features about students' views of relationships between figures and the students' responses were able to be interpreted within the SOLO model.

Introduction

The van Hiele theory (van Hiele, 1986), hypothesises five levels of thinking in Geometry which provide a framework from which to view students' understanding. Thinking at the third level is characterised by a focus on relationships that exist between figures and those that exist between properties. The notion of class inclusion is an essential characteristic of this stage of cognitive growth. Van Hiele (1986, p.95) described this aspect as requiring a student to "build up a network of relations in which the figures are interconnected on the basis of their properties".

Through researchers' (Mayberry, 1981; Usiskin, 1982; Fuys, Geddes & Tischler, 1985; Burger & Shaughnessy, 1986) explorations of the nature of the levels and the properties associated with the level framework, a more detailed and workable characterisation of Level 3 has emerged. Studies have acknowledged that class inclusion ideas are difficult to accept and simple questions requiring a yes or no answer, such as: Is a square a rectangle?, provides little insight into the thinking of students in this area. Some interesting findings, such as those of Burger and Shaughnessy (1986) and Pegg (1992), suggest that at Level 2, students precluded class inclusion classifications due to their own description of figures.

What is needed is a more intensive consideration of how students respond to activities which involve class inclusion ideas. This paper reports the results of one aspect of a larger study developed to explore these ideas. Of interest are the responses students made in an interview situation to an extensive exploration of how they linked (or saw connections among) different types of triangles.

To develop this, a task was designed which enabled students to return on several occasions to their response in order to further refine their ideas. To help with the analysis, the SOLO model was used to assist in the interpretation of the results once the categories of answers were identified. Before discussing the study, a brief overview of the SOLO model is provided.

The SOLO Taxonomy grew from Biggs and Collis'(1982) desire to explore and describe students' understanding in the light of the criticisms to the work of Piaget. Rather than focus on the level of thinking of students, their work focused upon the structure of students' responses. The framework has two main components, these being: the modes of functioning; and, the cycle of levels within each of the modes.

There are two modes of functioning relevant to this paper, namely, concrete symbolic, and formal. The concrete symbolic (C.S.) mode involves the application and use of a system of symbols which can be related to real-world experiences. This is the most common mode which is addressed in primary and secondary education. The formal mode involves the consideration of more abstract concepts. There is no longer a need for a concrete referent, and the person considers abstract or theoretical perspectives. Within these two modes there exist a cycle of three levels referred to as unistructural, multistructural, and relational. The sequence refers to an hierarchical increase in the structured complexity of a response regardless of the mode that is being targeted. General descriptions of the levels are:

Unistructural	- response is characterised by the focus on a single aspect.
Multistructural	- response is characterised by the focus on more than one independent aspect.
Relational	- response is characterised by the focus on the integration of the relevant components. The relationships between the known aspects are evident with consistency within this system.

Design

Three research questions guided the study. They are:

1. What was the nature of the links students formed when grouping different triangles?
2. Was there evidence of some developmental pattern in the different responses?
3. Does the SOLO Taxonomy provide a theoretical framework from which to explain the findings?

Twenty-four students, six from each of Year 8 to Year 11 (ages 13 to 17), were selected from two secondary schools. The students were of above average ability and there were equal numbers of males and females.

The nature of this study was to have the students identify and justify relationships among seven different triangle types, namely, acute scalene, obtuse scalene, right scalene, acute isosceles, obtuse isosceles, right isosceles, and equilateral. The format of the interview is contained in Table 1 and includes student tasks and the questioning focus common to each interview. This format was chosen as it enabled students to work with familiar recalled information, supplemented information, individual tree designs, and discussion involving prompts and probes from the interviewer. The continual revisiting of the same relationships, as drawn on different maps and discussed by the students, provided a vehicle for extracting further information, as the maps were used as a catalyst for discussion concerning the reasons for the existence of connections (or relationships). The analysis of the responses was facilitated by the development of a diagrammatical summary which combines the information gathered from student maps and interview transcripts. Four of these are included in the result section to assist in the interpretation of the findings.

Triangle Relationships

(i) Int: I would like you to write a list of all the triangle names you can think of. Begin with acute angled scalene. Draw each triangle.

(ii) Int: Design a tree diagram which links the different triangles. Draw a sketch to link each type.
(discussion follows concerning the reasons for links and/or lack of links)
(the following three points are addressed if required)

(iii) Int: There are some triangles that we can add to this list. (provide triangles not recalled)
Draw a sketch of each new triangle.

(iv) Int: Design a second tree diagram incorporating all the triangles on the list.
(discussion follows concerning the reasons for links and/or lack of links)

(v) Int: Return now to your first map. I would like you to add the new triangles to your original tree. (discussion follows concerning the reasons for links and/or lack of links)

Table 1: Summary Interview Structure

Results

Overall, the students found the ideas familiar, but the questions were seen as non-routine. The codings take into consideration the types of relationships described and the justification of these relationships. The students' responses can be summarised into seven types:

Type A A single similar feature is identified to relate triangles. Triangles appear in more than one group depending upon the identifying feature for each group.

Type B Scalene, isosceles and equilateral classes of triangles are formed and each is characterised by name and related by similar properties. No connection is made between the isosceles and equilateral classes of triangles. Sub-categories are formed with the addition of angle-type links across the isosceles and scalene classes of triangles.

Type C Three triangle-type classes are formed with angle-type links across the classes as with Type B. Similar properties are noted between the isosceles and equilateral triangles but the differences described do not allow a connection to be made.

Type D Relationships are formed across the equilateral and isosceles classes of triangles based on similar properties, but the equilateral triangle is not regarded as a subset of the isosceles class. Again, sub-categories within this type occur based on the addition of angle-type links across the three classes of triangles.

Type E Similar to a Type D response with indecisive statements concerning the possible inclusion of the equilateral triangle within the isosceles class of triangles, or statements concerning this notion of class inclusion without justification.

Type F The equilateral triangle is a subset of the isosceles class of triangles with justification based on properties.

Type G Similar to Type F but the subsets formed acquire further conditions. The relationship between the acute angled isosceles and the equilateral triangle becomes significant and can be justified.

To illustrate these categories and their SOLO interpretation, Types A, B, D, and F are considered with diagrams and brief relevant samples taken from the interview transcript. In the light of the SOLO Taxonomy, all the responses, apart from Group G, fall within the concrete symbolic mode. This occurs because they include the expression of concepts which are drawn from the context of the students' experienced world. In geometry, this means that the focus of the student's reasoning is primarily on properties that can be triggered by reference to a diagram.

Type A This type of response indicates the use of a number of single similar features or properties to relate triangles together, such as containing acute angles, unequal sides, or equal sides. Only one feature or property is used in each grouping, and the groupings change according to the property or feature which is the focus. Hence, a class of shapes has not formed an identity of its own. There was only one response coded as Type A and this is summarised in Figure 1 below.

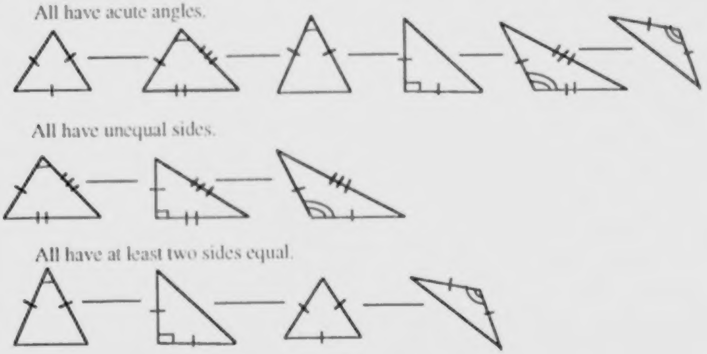

Figure 1: William's Triangle Relationships Summary

The following excerpt conveys the justifications for the relationships based on similar features or properties described by William. It illustrates that the relationships form spontaneously, and groupings change as often as the identifying property or feature changes. There were no links formed across the groups unless the student was prompted.

Int: *What have you done up on this top row?*
William: *They all have three sides and they all have at least one angle that is an acute angle.*
Int: *What have you done on the next row?*
William: *Um they all have uneven sides.*
Int: *And here?*
William: *They have, all have, at least two sides the same.*

This group of responses has a high visual element as relationships between triangles are based on a number of single observable features. Triangle groupings form spontaneously as a result of the identification of the relating feature with a strong reliance on visual cues. Due to the spontaneous nature of the relationships, classes of triangles have not formed generic categories. Lack of consistency is evident in groupings as the description of the relating feature is sometimes not applicable to all triangles of the group, e.g., in Figure 1, William's group of acute-angled triangles is based on the existence of at least one acute angle. In terms of SOLO this response would be seen as a transition response to the concrete symbolic mode.

Type B These responses include the formation of three mutually exclusive classes of triangles, these being, scalene, isosceles, and equilateral. Each of these classes represents a unit which has a specific name to encapsulate the similarities of the group. The similarities include one, or a combination, of the following; side properties, angle properties, axes of symmetry. None of the eight responses makes a connection between the class of isosceles triangles and the equilateral triangle.

Ellen's response (see Figure 2) represents the best of a Type B response as it includes the three classes (standard for this Type) and links that exist due to angle type. Overall, Ellen's justifications for classes of shapes are based on the following similar properties: (i) the equilateral triangle has three sides and three angles equal; (ii) the isosceles class of triangles has two sides equal, two angles equal, and one line of symmetry; and (iii) the scalene triangles have no sides equal and no angles equal.

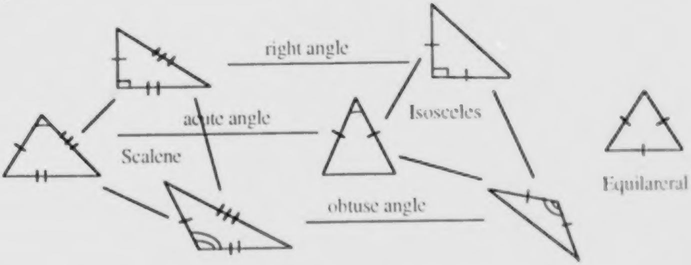

Figure 2: Ellen's Triangle Relationships Summary

A link is not made from the isosceles class to the equilateral class based on acute angles as the equilateral is described as having specifically three angles equal and therefore it is not possible to link the classes for any reason. The links across classes are only made according to angle types with the exception of the equilateral. This is illustrated by the following excerpt.

Ellen: *I have put the isosceles together.*
Int: *Why can they be linked?*
Ellen: *Because they have both got two sides the same. I have put these two scalenes together.*
Int: *Does it matter that one is acute and one is obtuse?*
Ellen: *Well they are still both scalene triangles. That one (equilateral) there won't link with any of them. It can only link because it has got three sides ...*
Int: *Can you think of a reason why this equilateral can't link up with these others?*
Ellen: *Um because it has got all equal sides and it is the only one.*

These responses represent a unistructural response in the concrete symbolic mode as each indicates that the described features/properties combine to form three classes of triangles, namely, scalene, equilateral, and isosceles. A class represents an identifiable unit, where any links are based on negative instances such as 'has two sides equal and one different' excluding the formation of subsets. Some responses include angle-type links across the isosceles and scalene classes of triangles.

The Type C responses are similar to the unistructural response above and represent a transitional group, as there is a tentative link between the equilateral triangle and the isosceles class of triangles. With prompting, a connection between these triangles based on a similar property is suggested, but not fully accepted. Reference is made to a connection between the equilateral and isosceles triangles, but the differences described prevent any connection being formed.

Type D This group of responses made links between the equilateral and isosceles triangle classes based on similar properties. Each of the eight responses in this group include three triangle-type classes with the addition of a link between the isosceles and equilateral classes. The relationship is based on both classes containing two equal sides and/or two equal angles, although, the equilateral triangle is not yet described as a subset of the isosceles class of triangles.

Cameron's response as summarised in Figure 3 is typical of a Type D response including three angle-type links. He described the triangle-type classes in terms of similar side properties. The link between the equilateral and the isosceles class of triangles exists on the basis of both classes containing two sides the same length.

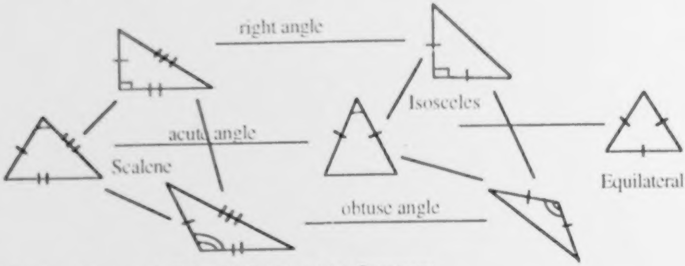

Figure 4: Cameron's Triangle Relationships Summary

Int: *Can you tell me about the link you have made between the equilateral and the isosceles?*
Cameron: *The equilaterals link because they have both got two sides that are equal and two angles that are equal, the isosceles have one side and angle that is not going to be the same.*

These responses represent a multistructural response in the concrete symbolic mode as the properties describe the figures, and the figure is known by its properties. Unlike the unistructural response where the differences dominated the similarities, here both can be dealt with separately. Inconsistency, however, lies in the inability to consider in the equilateral triangle that two sides and two angles equal is a subset of three sides and three angles being equal. Sub-categories are also evident due to the addition of various angle-type links.

Type E can be described as transitional as they are characteristic of a multistructural response with the addition of tentative statements concerning the equilateral triangle as a subset of the isosceles class of triangles. When prompted, these responses describe the notion of class inclusion as a possibility but without acceptance, or initiate this notion but without justification of the relationship.

Type F A typical response describes the isosceles class of triangles as containing the equilateral triangle. The equilateral triangle is identified as a form of the isosceles triangle, and the student is able to justify the equilateral triangle's existence within this class. It can also be argued why an equilateral triangle is an isosceles triangle and the opposite is not the case. There was one response coded as Type F. The diagram developed from the information given by Sally is contained in Figure 3.

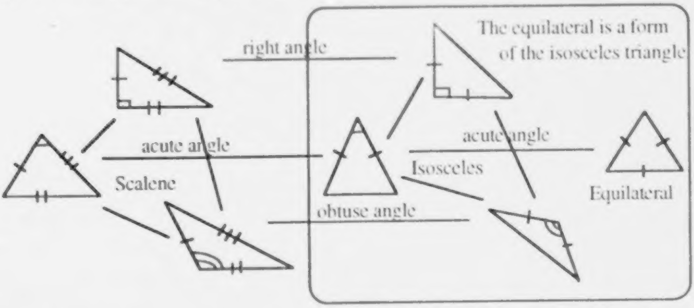

Figure 3: Sally's Triangle Relationships Summary

Sally's explanation of the tree diagram illustrates the clearly defined relationships that exist between the classes of triangles. Sally included the notion of class inclusion as an important feature of her tree design and was able to justify this on the basis of properties.

Sally: *Um, all the triangles begin the tree. Then I differentiate between the side length with two or more equal sides and sides are not equal.*

Int: *So you have ended up with the equilateral and the isosceles on the same branch. Do you see those two triangles linking?*

Sally: *In that they have equal sides and equal angles. You could say that the equilateral triangle is a form of the isosceles triangle in that it does have two equal angles and two equal sides.*

This response is categorised as relational in the concrete symbolic mode as it includes the notion of class inclusion as the integrating feature of the described relationships. The equilateral triangle is considered a subset of the isosceles class of triangles, thus, illustrating consistency in terms of the described relating features of the triangle classes. The equilateral triangle, and the scalene and isosceles classes of triangles each maintain a workable identity which takes into consideration the network of relationships based upon the properties of each class.

A Type G response accepts the class inclusion concept identified in the previous level but brings into the discussion the constraints that need to be imposed on the isosceles triangles in this task. The equilateral triangle is described as a subset of the isosceles class of triangles with a significant link to the acute isosceles triangle. This differs from earlier attempts when only the visual feature of angles being acute were acknowledged.

Conclusion

By considering the structure of the responses, a hierarchical framework has emerged which sheds light on the development of student understandings of triangle relationships. An important implication of the identification of different types of responses which all incorporate connections between triangles is highlighted by the justification, and in some cases the absence, of links. While connections are made between the same triangles by different students, the nature of these vary. For example, when considering the connection between the equilateral triangle and the isosceles class of triangles they range from: precluding any connection because of the closed definition of each triangle group, recognising two sides or two angles equal without acceptance of the equilateral triangle as a subset of the isosceles class of triangle, and finally, recognising the equilateral triangle as a subset of the class of isosceles triangles.

A developmental path was identified which leads to class inclusion as earlier notions were subsumed by later ideas. For example, the simple aspects originally used to identify a particular class of triangles which are independent and isolated, were later used to relate the classes together, and, finally, used to justify the inclusion of different types of triangles into a category. In addition, transitional responses were noted under probing where students were striving to provide a more comprehensive response.

The SOLO model proved valuable in being able to assist in the interpretation of the developmental pattern identified. The unistructural response (C.S. mode) is where the triangle classes, scalene, isosceles, and equilateral, take on an independent meaning, and become a viable element in their own right. Any links between the classes were based on visual features only. The multistructural response (C.S. mode) marked the emergence of the properties as a viable aspect which were able to be employed in more than one class. This comparison meant that the equilateral triangle was seen to have two sides and/or angles equal, however, this was not sufficient for students to consider the equilateral triangle to be isosceles as the equilateral triangle had three sides and/or angles equal. The relational response (C.S. mode) indicated that the student was able to rationalise the logic of both three sides and/or angles being equal, and two sides and/or angles being equal within the one triangle. It is the ability to hold all the relevant elements associated with the equilateral triangle, and at the same time consider possible subsets which leads to the realisation that the equilateral triangle is isosceles.

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"MAKING SENSE OF SINE AND COSINE FUNCTIONS THROUGH ALTERNATIVE APPROACHES: COMPUTER AND 'EXPERIMENTAL WORLD' CONTEXTS"

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This paper summarises the main results of a master dissertation which has investigated the influence of two different contexts — computer and "experimental world" — on trigonometry learning identifing which introduction order would be more effective. A didactic sequence was prepared to be applied into two groups of high school students: the Group B was first introduced in computer activities followed by 'experimental world' manipulation, whilst the inverted order was done to group C. There was also a reference group A. The results showed that introduction order has indeed interferred in learning process.

Introduction

The difficulty of learning trigonometrical functions has been enphatically stressed by students throughout our teacher careers. Mathematics teachers also find hard to help students make sense of this topic. However very few research has been made exploring this matter, mainly with specifically regarding the learning process of sine and cosine functions.

Looking through recent research it is possible to note the increase of studies based on constructivism point of view which has been used either computer (Wenzelburger, 1992; Gomes-Ferreira, 1997; Borba, 1993; Hoyles, 1991, 1996) or everyday life (Saxe, 1991; Lave, 1989; Nunes, 1993, 1996) or even both contexts (Magina, 1994).

Our concerning was to have both above contexts as a mediator tool in learning process of trigonometrical functions. We had as hipothesis that both contexts would certainly be suitable for our proposal and moreover they would complement each other. However we could not predict wether the introduction order of these contexts would interfere on the formation and development of student's trigonometry concept.

The theoretical background to support this study came from a triplet of Vygotsky's ideas — zone of proximal development and indeed the role of cultural tool as a way to make a bridge between individual and mathematical concepts —, Vergnaud's thought — theory of conceptual field, understanding that formation concept emerges from problem solving where students are asked to deal with a several number of different situations which reffer to related concepts — and Piaget's viewpoint, who explicitly states that knowledge is formed by two aspects: figurative and operative and these aspects appear from the development of symbolic function. We could not consider Piaget's ideas without understanding them inside the equilibration process which is the central point to acquire and develop knowledge.

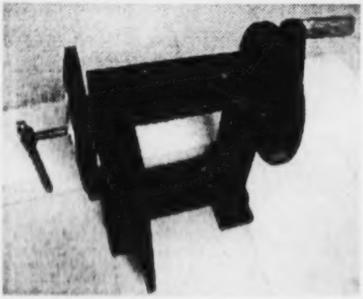
We indeed gave great importance for the role the contexts would play in our study. Context has at least four different meanings in Mathematics Education. The first is "what comes with the text" (con-text), which states that all written texts include explicit or implicit informations and the meaning of any text depends on subject reading experience (It is the case of Geertz, 1973 and Rorty, 1989 works). The second meaning is related to real-world phenomenon that can be modelled by educators in order to give sense to mathematical concepts (which is exemplified by Magina,1994 and Confrey, 1991 works). The third meaning for context is "semantic situation", i.e., where students can associate the activity with their everyday life (as Nunes et al, 1993). The last way of understanding context is as a setting, it is related to place or physical site of human activities, such as scientific laboratory, a supermarket, a computer laboratory, a factory, etc (as Saxe, 1991). We used context as refering to the last meaning. However the 'semantic situation' was also embeded in it. In fact, whatsoever the meaning of context, there is a consensus among Mathematics Educators that is quite impossible to deal with any mathematical contents without considering both influence of context and situation in which students are going to experience this content.

We opted to work with sine and cosine trigonometrical functions, for this proposal we planned a didactic sequence whose aim was to introduce the concept of sine and cosine functions and their transformations, i.e., functions such as $f(x) = a \text{ sen } (\omega x + x_0) + b$, or $f(x) = a \cos (\omega x + x_0) + b$ with a, b, ω, x_0, real numbers, $\omega > 0$ and $a \neq 0$. This sequence involved two different contexts, one of them was 'experimental world' where students were asked to solve problems by dealing with concrete materials such as wood, sand, metal, etc. We chose to build activities from physics phenomena which explored Uniform Circular Movement and Simple Pendulum Movement. The reason for our choice was based on our belief that those physical phenomena make the visualization of periodic movement easy for students' perception. Three activities were planned in this context: "Optic Alarm Simulative" built from a kit composed by an analogic watch and two lamps (see Fig.1), "Wheel with Laser Pen" (see Fig.2) and "Sand Pendulum" (see Fig.3). Another context was the computer where activities were developed using "Cabri Géomètre II" and "Graphmatica for Windows" Softwares. Although both contexts were being considered as alternative ones, they were clearly distinguished from each other. Whilst activities embebed in 'experimental world' were characterized by informalism and intending to represent (modelling) the real world, the computer context showed as the main characteristic the immediate feedback, the possibility to improve movement as well as to draw a lot of figures quickly and precisely.

This model was built with two lamps which illuminated a watch. All hands of this watch were took out from it and a shaft was put instead of the second pointer, in order to make a shadow in a milimetric paper. In this way a trigonometrical point in motion with its projection sine and cosine could be simulated. Students were asked to study this model to relate each point of the cycle with its projection, in order to solve an espionage problem.

Beginning with this problem we introduce sine and cosine functions.

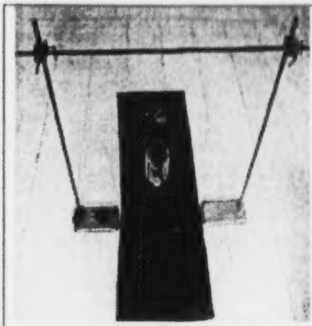
FIG. 1 : Optic Alarm Simulative



This model was a wood equipment composed by two wheels. In one of them there was a handle and in the other one there was a laser pen fixed which made a point of light up the table.

Students were asked to turn round the wheel while pushing the whole equipment to the left hand side. Afterwards they shoud describe the trail of the point of light for another student who had to draw it.

FIG. 2 : Wheel with Laser Pen



The equipment was a Pendulum made by a plastic bottle filled with sand. The bottle was fixed to a metal shaft by a string. Below the Pendulum there was a paper which was to be pulled by a student when the pendulum started to move and drop out sand.

Students were asked to predict which figure would be drawn in the paper. After this, they should test their prediction by manipulating the Pendulum.

FIG. 3 : Sand Pendulum

The Study

The research involved 32 high school students from a private school in São Paulo State, Brazil, who were divided into three groups: A, B and C. Group A was the 'reference group', while the other two formed the experimental groups. The study comprised five phases.

Phase 1 - PRE-TEST: It consisted of a Paper & Pencil test composed by eight questions divided into 10 items following the formal way of school tests, i.e. seven out of the eight questions were presented descontextualizedly. The test was applied collectivelly to each group but students solved it individually. The aim of this phase was to investigate the amount students could respond in a trigonometry test before getting involved in our didactic sequence.

Phase 2 - DIDACTIC SEQUENCE: For Group B the activities of the didactic sequence were developed in the 'experimental world' context whilst for Group C the activities were inserted in the computer context. This phase was developed simultaneously for each group and took the same number of sessions. Students from Group A had trigonometry subject in their class taught by the school teacher (not the researcher). This group had three classes aiming the introduction of sine and cosine trigonometrical functions in oral explanation as well as by doing exercices.

Phase 3 - INTERMEDIATE TEST: Such as the previous test, this was also elaborated following the tradicional school test. Although this test covered the same trigonometry topics as the Pre Test, it involved only half number of items. The reason for having fewer questions was to reduce students tiredness. The main goal of this test was to observe whether the groups presented different progress in understanding trigonometrical concepts. In this way we could evaluate the influence of each context on our sample.

Phase 4 - DIDACTIC SEQUENCE: Here groups B and C changed contexts, i. e., Group B worked inside computer context while Group C in 'experimental world'. The Group A continued to have trigonometrical functions in classroom.

Phase 5 - POST-TEST: It was applied as the same as the other two tests. Similarly to the Pre-Test it involved the same number of questions and items.

The table below is to show the contents involved in each questions and also the relationship among the tests.

Item	Pre Test	Intermediate	Post Test	Contents by items
1	Question 1a	Question 3a	Question 6a	Comparing and completing with >,<,= (sine)
2	Question 1b	Question 3b	Question 6b	Comparing and completing with >,<,= (cosine)
3	Question 2a		Question 5a	Finding two valuers for q taken sen q
4	Question 2b		Question 5b	Finding two valuers for q taken cos q
5	Question 3		Question 4	Completing with function's maximum and minimum value
6	Question 4		Question 7	Given the function's image find the "a" parameter
7	Question 5		Question 1	Given a function's graphic, choose the correct algebraic expression
8	Question 6	Question 2	Question 2	Using the Fundamental Trigonometric Relation
9	Question 7	Question 1	Question 3	A contextualized test in physics
10	Question 8	Question 5	Question 8	Link the graphical shape with algebraic expression

Table 1: Topics included in the three tests distributed into ten questions

The follow diagram is to present a summary of the research design

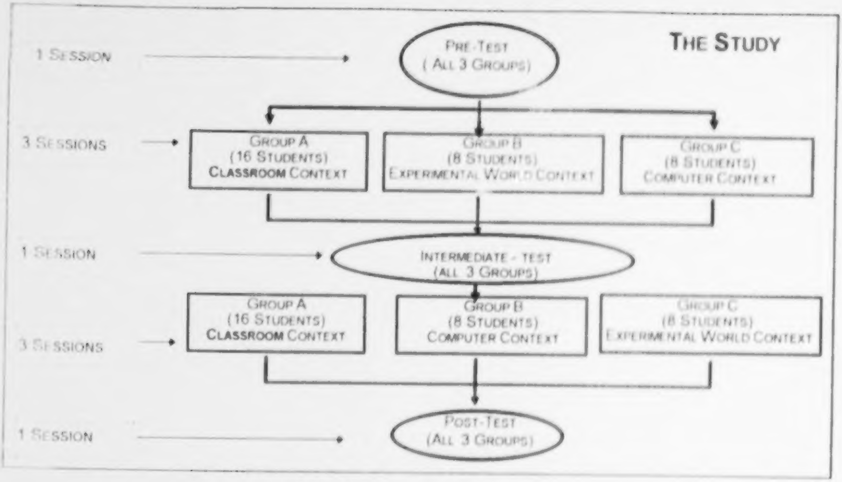

THE STUDY
1 SESSION
PRE-TEST
(ALL 3 GROUPS)
3 SESSIONS
GROUP A
(16 STUDENTS)
CLASSROOM CONTEXT
GROUP B
(8 STUDENTS)
EXPERIMENTAL WORLD CONTEXT
GROUP C
(8 STUDENTS)
COMPUTER CONTEXT
1 SESSION
INTERMEDIATE - TEST
(ALL 3 GROUPS)
3 SESSIONS
GROUP A
(16 STUDENTS)
CLASSROOM CONTEXT
GROUP B
(8 STUDENTS)
COMPUTER CONTEXT
GROUP C
(8 STUDENTS)
EXPERIMENTAL WORLD CONTEXT
1 SESSION
POST-TEST
(ALL 3 GROUPS)

Diagram 1 : The Experimental Design

Results

The way we chose to analyse the efficiency of each context was by using formal tests. We decided to evaluate the contexts through paper & pencil test because it is the school context by excellence, i.e., students are used to answering written questions using paper and pencil. We also consider that if the evaluation test took place either in computer or 'experimental world' contexts we would be privileging those students who took part in our study and in this way no comparison could be made between the experimental groups and the reference group. In addition, we tried to avoid having an evaluation strictly related to the contexts involved in our didactic sequence. This was supposed to give us a chance to observe if and the amount students were able to transfer knowledge acquired, after they had carried out activities inside both computer and "experimental world" alternative contexts, to the school context (formal context).

Having this in mind, we analysed the results through seven different viewpoints:

a) The general performance of the all groups in the three tests by looking at percentage of correct answers in each group;
b) The variation tax of correct answers obtained in each groups considering Pre and Post Tests;

c) The individual performances of the experimental groups considering the percentual variation tax of each student correct answer from Pre to Post-test;

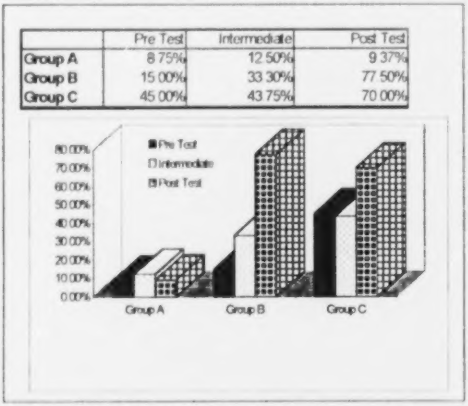
d) The percentage of correct answers considering the didactic sequence aims;

e) The number of correct answers in the items, looking at Pre and Post test results;

f) The variation tax of correct answers in each item;

g) The types of mistakes and procedures made by the experimental groups, taking into consideration both Pre and Post-tests.

For the purpose of this paper and also considering the space we have to present our results, we have decided to discuss only three out of these seven viewpoints (a, b and e viewpoints) because we believe that they were powerful enough to show an overall of the three groups performances. All tables are presentied beside their respective graph. The **Table 1** is to show a general picture of all three groups with regarding Pre, Intermediate and Post Tests. **Table 2** shows the variation tax of correct answers of the three groups considering the Pre and Post Tests. Finally **Table 3** refers to the number of correct answers obtained from each experimental group taking into account the items contained in Pre and Post Tests. The discussion will start by looking at the tables related to these viewpoints.



	Pre Test	Intermediate	Post Test
Group A	8.75%	12.50%	9.37%
Group B	15.00%	33.30%	77.50%
Group C	45.00%	43.75%	70.00%

Table 2: General Group's Performance

From **Table 1** it is possible to interpret that the didactic sequence an effective way to introduce trigonometrical functions to students.

It is clear that experimental groups performed better than reference group.

Looking at percentages of correct answers of group B and C in the Post Test we could say that they were closed to each other.

However the performance of group B was more consistently increased.

Table 1 also indicates that both experimental groups presented greater leap of correct answer from the Intermediate to Post Test than from Pre to Intermediate test.

Regarding the groups variation tax from Pre and Post Test, (see **Table 2**) group B presented an enormous percentage of variation in comparison with the other two groups.

	Pre Test	Post Test	Variation
Group A	14	15	0 62%
Group B	12	64	65%
Group C	36	56	25%

OBS The maximon number of correct answers for Group A in each test is 160 (i e 16 students by 10 itens)

For Group B and C are 80 (8 students by 10 itens)

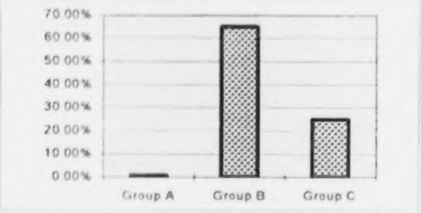


Table 3: Tax of Correct Answers Variation

If we consider that a test can be a thermometer which expresses how much a person knows about something, **Table 2** points out that the school method was not effective enough to develop the introductory trigonometrical concepts among these students. In fact added the information given by Tables 1 and 2, the reference group started from a low level of knowledge, increased a little bit during a trigonometry class and thus came back to almost the same point as the beginning.

Looking at the experimental groups performance from items perspective, **Table 3** shows that group B obtained better results than group C in 60% of the items (six out of 10 itens), and only in 20% of them (two out of 10 itens) group C was better than B.

Group B		
Item	Pre Test	Post Test
1	2	7
2	1	6
3	1	7
4	1	5
5	0	6
6	0	8
7	2	8
8	2	5
9	0	3
10	3	7

Group C		
Item	Pre Test	Post Test
1	5	6
2	2	4
3	5	5
4	6	5
5	0	6
6	2	7
7	5	6
8	7	8
9	1	4
10	2	5

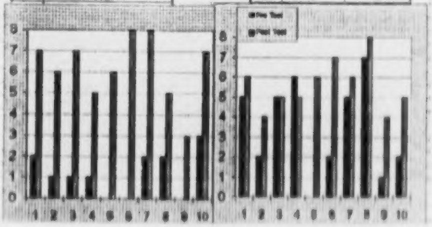


Table 4: Number of Correst Answer by Items

In fact, looking at items 5 and 6 of Pre Test there was no student from group B able to answer them correctly but after the sequence this number increased to 6 and 8 students respectively. Moreover in almost all itens (except itens 8 and 9) the number of students who correctly answered them was increased.

We concluded that both group performed better from Pre to Post Test, however group B presented the best results.

Conclusion

The general analysis of a students'performance which participated in our experiment pointed to a constant growth of formation and development of concepts, to both experimental groups, from the significative increase in the number of correct answers in Pre and Post test. The application of the didactic sequence showed how profitable it was to work in two contexts, since we noticed throughout the development of group study that students were able to make correspondences among the presented tasks in each context. Especially Group B - students have several times referred to the facts which were observed in 'experimental world'when working with the computers tasks. We are convinced that in order to develop the approached subject, according to our didactic sequence, both contexts were necessary and complementary.

The main experiment conclusion supported by our analyses is that in this research, the introduction order of contexts interfered on learning process. Whatever the viewpoint we chose to observe the students'groups and the most successful was the one which worked with 'experimental world' and afterwards with the computer (Group B).

Our research suggests that the learning process in computer contexts becomes more efficient when students do not have any previous contact with the subject and when this subject is preceded by concrete manipulation in less compromised to formal situations.

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TEACHER AND STUDENTS FLEXIBLE THINKING IN MATHEMATICS: SOME RELATIONS[1]

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ABSTRACT: *We examine the characteristics of the successful/unsuccessful student in Mathematics considering psychological and cognitive factors. The successful student at a long range was identified with that one which exhibits a flexible way of thinking, which allows him to attain a more global and significant understanding of the concepts. Starting from teacher and students interactions and from classroom observations we have verified that when teachers make use of flexible forms of thinking, they may be contributing for some students to seek for a sense in the use of concepts and formulas, and also contributing for their success in Mathematics. We suggest that teachers deliberately assume a classroom attitude which will stimulate problem solving abilities and forms of flexible and autonomous thinking.*

Flexibility of thinking and success and failure in mathematics

In our experience, we have noticed that the teaching process might be contributing for error and failure in mathematics. The emphasis in formal procedures (algorithms and rules) unrelated to the concept that supports them inhibits the necessary flexibility of thought that is essential for success in mathematics (DAVID & MACHADO, 1996).

GRAY & TALL (1993) dealt with the question of success and failure in mathematics going into the way students work with certain mathematical concepts: the **procepts**. To these authors the **procepts** are mental objects that consist of a certain combination of a process and a concept produced by that process, and a symbol that may be used to represent either or both of the above. For instance, the symbol $3+2$ can be either the process of adding 3 to 2 or the representation of the concept of addition of 3 to 2. According to these authors the notion of **procept** applies to those concepts in arithmetic, algebra and analysis which are initially learned through a process, but not to concepts learned by definitions or to the major part of geometrical concepts which are introduced through visual perception.

GRAY and TALL claim that students with success in mathematics are those that master the mathematical symbolism both dealing with the symbol as a process and realizing that underlying the symbol there is a mathematical concept, as in the

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[1] This research report is partly supported by CNPq and has received colaboration from the students Gizelle da Silva Leite, Alisson Augusto Marques and Denise da Silva Ribas Capuchinho

example above. These students can more easily both establish mathematical connections and draw generalizations.

The work of GRAY and TALL contributed towards helping us understand and make more explicit some problems in the teaching of mathematics which we have been thinking about for some time now. It was a starting point in our interest for the didactic-pedagogical factors that can lead the student to develop a greater flexibility of thinking that enables him/her to reach a more global understanding of the concepts. This understanding assumes the mastery of the symbolic language specific to mathematics and the association between the symbol and a process or a concept represented by the symbol or with other , more general, concepts related to it as the situation arises. The flexibility of thinking is related with the easiness with which the student is able to cross these different levels of association in the adequate moments.

Research Methodology

We are developing our work in the Brazilian school system. We recognize its assessment system privileges those students coming from the ruling classes and that it has a conception of success and failure that is strongly related to social factors.

Despite our awareness of this situation and of the importance of other factors, such as those of psycho-social and psychoanalytical nature and those related to the structure and organization of the school system, we concentrate in our work, nevertheless, in the analysis of cognitive factors keeping it in the pedagogical level, being given our professional expertise.

Despite the fact that the aspects being considered here are seen by various authors as less important to the question of success/failure in school than the aspects referred to above, our aim is to demonstrate that the analysis of the interaction teacher-student-content, made from the point of view of the classroom, may also give a significant contribution to the discussion of the question being envisaged.

It is important to recall here the group of researchers that have been working in the line of research of 'cognitive acceleration' and 'teaching to think' (ADEY, 1988; COLES, 1993; McGUINNESS, 1993; TANNER & JONES, 1995). Their work has been demonstrating that teacher's interference can play a significant role in the development of their student's thinking. These researchers base their work in instruction models that start from the idea that the thinking and learning processes are social constructions adopting a socio-constructivist approach.

We interviewed 21 students, some considered to be "good" students, others "mediocre" and still others "weak" in Mathematics. The students belong to 5th, 7th and 8th grades. They were observed during a month with an average of four lectures being observed each week, in each class. We followed this classroom activity with the aim of testing the method employed in the research. Considering the objective of our work is to identify teaching methods that could be contributing for students' flexible thinking, we chose an observational and interpretative approach, i.e., direct observation of the classroom followed by an analysis of the observed data. In what follows we give only a short account of the interviews and observations we have made; a more complete account can be found in DAVID & LOPES (1997).

We chose this school because its students "are known" for showing an independent way of thinking and for having an attitude of great autonomy regarding the learning process. We expected to find in this school teachers with methods of instruction that would bring forward flexible ways of thinking , that is, emphasizing, among other things, those abilities of thought which are stressed by the cognitive acceleration research group (COLES, 1993; TANNER & JONES, 1995). That is to say, we thought we might find in this school teachers that would encourage their students to think and plan their own work by themselves as well as discuss it.

The students' point of view about success and failure in mathematics

The interviews carried out in this school stress the association, on the one hand, between the "good student" and the diligent, attentive and interested one, and on the other, between the "weak student" and the inattentive one, that does not do properly his/her work.

We asked the teachers that participated in our research to name, in each classroom, two or three pupils that they would classify as "good in mathematics" and two or three that they would rate as "in the mean", so as to be interviewed.

These interviews aimed at deepening our classroom observations and at getting to know facts related to the student's school life that would allow us to identify striking influences in his/her relation with mathematics.

Among the students interviewed those considered "good" maintained they have always had interest in mathematics, easy understanding and pleasure in the subject. The students considered to be "weak" stated that they had always felt difficulties in the subject and that they did not like neither the subject nor to work in mathematics. Generally speaking they did not associate their difficulties in the subject with the teacher and his/her attitudes in the classroom.

In the group of students we interviewed, we found three which were very close to what is, in our conception, a good student in mathematics, i.e., a student who makes use of logic to solve problems free from given formulas and pre-established solving strategies. They were the ones who were closer to the idea of flexibility of thought (GRAY & TALL, 1993): ability to establish relations between different concepts and to move freely between the concept its symbolic representation and usage. In particular, among all the students observed, student Gu8[2] , is the student that is closest to GRAY & TALL's "successful student". He showed independent ways of problem solving which for the considered situations were more adequate, as they were simpler and related to subjects of study learned before. This student's procedures are opposed to those pupils' procedures imprisoned by canonical methods of thinking, rules and formulas. The above mentioned authors stress that those latter students obtain a short lived success. In the long run they are bound to fail

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[2] Students will be represented by an abbreviation of their names followed by the number 5, 7 or 8, according to his/her grade.

Teachers that make explicit the employment of flexible forms of thinking might be contributing to the success in mathematics of some of their pupils

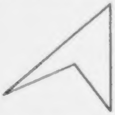
At first sight, the observations made in the classroom did not match our expectations. We found situations in which the instructors seemed to lead to a dialogue with their students that could encourage them to act autonomously so that they themselves would take in their hands the learning process. Immediately, however, they would go back on their own steps and take a posture that would inhibit this autonomy.

The example of teacher's interference that we considered the most positive from the point of view of contributing to the flexibility of thinking of the student appeared in a geometry class. It concerns the sum of the interior angles of a parallelogram. The teacher asks: *'What if the parallelogram has angles greater then 90 °, how can the total still add to 360 °?'* This question produces a discussion between two students where they search for a reason for the fact. *'They are not all greater than 90 °, they are not all equal.'* This example ends with a suggestive example presented by the teacher.

It can be noticed that there was a positive interference from the part of the teacher that draws an analogy with the sum of the interior angles of the rectangle that had been studied before.

Next the teacher challenges the creativity of student Je5. His aim was that the student would generalize the result about the sum of the interior angles of a parallelogram to any quadrilateral.

> *'The teacher asks Je5 to draw a very crazy quadrilateral.*
>
> *Je5 draws:*



> *The teacher praises the pupil's creativity and says it is a quadrilateral but not a convex one and that therefore it is not going to be studied at the moment. He asks the student to draw another one'.* (Classroom notes)

Je5 does as the teacher says drawing a convex, "less crazy", and less creative quadrilateral and the class goes on.

The teacher misses the opportunity of exploring a little bit more the study of quadrilaterals and its interior angles and ignores the possibility of making an interesting generalization about the sum of the interior angles of non-convex quadrilaterals and calls back the attention of his students to routine problems.

We found, nevertheless, in this school some students with the characteristics described by GRAY & TALL (1993), as in the following example:

> '*Exercise: Reduce to the same denominator* $\frac{3}{5}, \frac{2}{10}, \frac{4}{3}$.
>
> *The teacher develops the algorithm in the blackboard :*
>
> $$\begin{array}{c|c} 5,\ 10,\ 3 & 2 \\ 5,\ 5,\ 3 & 3 \\ 5,\ 5,\ 1 & 5 \\ \hline 1,\ 1,\ 1 & 30 \end{array}$$
>
> *And says that it means 30 is the least common multiple of 5, 10 and 3.*
>
> *He writes on the blackboard :* $\frac{}{30}, \frac{}{30}, \frac{}{30}$ *and asks for the numerator of the first fraction* $\frac{3}{5}$ *to be found.*
>
> *Ma5 interferes saying it should be 18 since the denominator 5 has been multiplied by 6, therefore the same should be done with the numerator.*
>
> *The teacher says it is enough to do* $(30 \div 5) \times 3$. ' (Classroom notes)

The interference of the student is based on the concept of equivalent fractions and he did not feel the need to evoke the algorithm while the teacher is attached to the canonical procedure.

Trying to understand these cases we decided to readdress our attention to the teachers with the aim of verifying , if the classes we watched, gave us ground to decide whether or not the teachers themselves were making use of the flexible way of thinking in their classes. Shifting thus our attention from the student to the teacher we went back to the classroom notes for a new reading. It allowed us to verify that even though the observed teachers were not encouraging in a totally deliberate and conscious way flexible ways of thinking among their students the teachers themselves seize upon flexible ways of thinking in several moments during their classes. Thus in an non-intentional way they end up serving as an example to their students that adopt some of these ways of thinking used by their teachers.

There are several examples of this kind of situation we could present here. We elected to show here two of the most significant ones.

Example 1

> '*A girl (Gi7) asked to solve an exercise but the teacher asked another student Th7 to solve it instead. Th7 should do:* $\frac{2}{x^3} + \frac{5}{x} + \frac{1}{x^2}$. *When Th7 put* x^3 *in the denominator, the other students were not sure whether it should be* x^3 *or* x^6. *He made an error and ended up with* $\frac{2x+5+x}{x^3}$.

> *Since they were unsure whether x^3 or x^6 was correct, the teacher showed that using different methods the answer turns out to be the same (...) Going on the teacher explained that with x^6 one would have to factor $\frac{2x^3+5x^5+x^4}{x^6}$ and she asked which factoring method should be used and the answer was the common factor method. Immediately after crossing out the x^3, the teacher showed that the answer would have been the same if the l.c.m. had been used.'* (Classroom notes).

The teacher explores the student's difficulties exchanging views with them and comparing the two solutions.

Example 2

Going back to the sum of interior angles of a quadrilateral class we noticed P5[3] encourages his students to make generalizations and to justify them.

The procedures used by this teacher are more frequently closer to that variety of procedures that is characteristic of the cognitive acceleration programs mentioned before.

As we discussed before, P5 encourages his students to make generalizations:

> *'P5: What is the sum of the angles in a quadrilateral?*
> *Some students answer 360 °, and Ma5 asks whether this is true for any quadrilateral. The teacher says it is and asks someone to provide a proof for the statement. Mrc5 interferes saying that it suffices to divide the quadrilateral in two triangles and that each triangle has 180 °. The teacher approves and draws the figure in the blackboard.'* (Classroom notes).

Although this discussion is happening in a fifth grade classroom we noticed a deliberate intention of P5 to carry his students into making an informal deduction.

Final Considerations

Summing up, these examples suggest that the very fact that the teacher himself seizes upon some characteristics of flexible thinking in his/her classes, could be contributing, even in a non deliberate way, to develop in his/her students some thinking abilities important for the learning of mathematics.

A similar point of view was presented by HIRABAYASHI & SHIGEMATSU in a work cited by TANNER & JONES:

> *'... Hirabayashi and Shigematsu (1987) argue that students develop their concepts of metacognition by copying their teacher's behaviour, and thus,*

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[3] Fifth's grade teacher.

> *their executive or control functions represent an "inner" teacher.'* (TANNER & JONES, 1994, p.420)

HIRABAYASHI & SHIGEMATSU discuss their results supported by VYGOTSKY's work. In fact, VYGOTSKY (1996) states that all superior psychological functions are originated from the real relations between individuals in a process which develops from the *interpersonal* to the *intrapersonal* relations, up to being definitively internalized.

This non-deliberate influence of the teacher needs to be deeply investigated, through a more systematic observation of the work developed in the classroom and through the study of the development of the students' thought.

Moreover, the point of view expressed by researchers such as COLES, who are discussing principles, problems and programs for the 'teaching thinking', suggests that there should be a more directed action of the teacher:

> *'...though it is reasonable to argue that good teachers in all subjects encourage their pupils to think, this is not the same as teaching them how to think, for instance by explicitly drawing their attention to the kind of thinking they are engaged in. Teaching thinking means providing not just encouragement and opportunity, but a knowledge of principles and techniques, and regular guided practice in applying those principles and techniques.'* (COLES, 1993, p.339)

Our analysis led us to believe that, if the teacher intentionally carries through a work in the direction of developing those abilities, he/she can reach a more significant number of students. He/she can even reach those students, which are the great majority, and sooner or later end overcome by failure in school mathematics.

Explaining in the interviews how they would solve mathematical problems, those students said they would try to remember rules, formulas or concepts learned before, or follow the model in the textbook or in the example given by the teacher. Thus, they were being trained to answer what the teacher is expecting, and they would not be acting autonomously. These students may even be well succeeded in school tasks without necessarily making use of a reflexive thinking. Nevertheless, as GRAY & TALL (1993), we also believe that these students' success in mathematics is only transitory. From a certain point onwards, either because the level of demands on the reflexive thinking increases, or because the level of demands regarding the amount of rules and algorithms to be trained and memorized increases, their study procedures fail to respond satisfactorily to these demands.

Our results suggest, therefore, that the abilities of thinking necessary for a long range success in mathematics are not being sufficiently encouraged by teachers. This situation seems particularly serious nowadays, when mathematics educators are diminishing the importance of techniques and algorithms, and pointing to the necessity of developing some 'basic' abilities (CARVALHO & SZTAJN, 1997; D' AMBRÓSIO, 1997; LELLIS & IMENES, 1994), almost always related to problem solving abilities which presupposes an autonomous and flexible thinking, in the same sense we have been discussing. This present perspective is practically a consensus and it can even worsen the problem of failure in mathematics, if teachers are not to quickly and deliberately adopt a classroom posture to encourage autonomy and flexibility. If from now on students are to be evaluated according to these abilities, even those which still attain some success today because they cope well with school tasks, will be faded to failure.

Our present research in other schools, still at an initial stage, seems to add further evidence to this point of view. It will enable us to provide more orientation for the teacher who is willing to contribute for his/her students' success in mathematics.

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THE INFLUENCE OF METACOGNITIVE AND VISUAL SCAFFOLDS ON THE PREDOMINANCE OF THE LINEAR MODEL

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Dirk Janssens (University of Leuven; Belgium)

Two recent studies by De Bock, Verschaffel & Janssens (1996, 1998) revealed a very strong tendency among 12-16-year old pupils to apply the linear model in non-linear problem situations involving length and area of similar plane figures. More recently, we executed a new investigation in which we manipulated two new aspects of the experimental context to get a better understanding of the predominance of the linear model observed in our previous studies, namely (1) the provision of a metacognitive and (2) of a visual scaffold. While both scaffolds yielded a significant effect on the number of correct answers on the non-linear problems, this effect remained remarkably small, suggesting that pupils' tendency towards linear modelling is indeed very strong, deep-rooted and resistant to change.

THEORETICAL AND EMPIRICAL BACKGROUND

Pupils' tendency to apply linear proportional reasoning in problem situations for which it is not suited, has been frequently described and illustrated in the literature on mathematics education (see, e.g., Berté, 1993; Freudenthal, 1983, Rouche, 1989), but until recently, empirical data on the scale and persistence of this phenomenon were practically absent. In two recent studies (De Bock, Verschaffel & Janssens, 1996, 1998), this phenomenon of unbridled proportional reasoning was amply documented for secondary school pupils working on word problems involving length and area of similar plane figures. In these two studies a paper-and-pencil test was administered to 120 12-13-year old and to 222 15-16-year old pupils. The test involved 12 experimental items about similar plane figures: 4 items about squares (S), 4 about circles (C), and 4 about irregular figures (I). Within each category of figures, there were 2 proportional items (e.g. "Farmer Gus needs approximately 4 days to dig a ditch around his square pasture with a side of 100 m. How many days would he need to dig a ditch around a square pasture with a side of 300 m?"), and 2 non-proportional items (e.g. "Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?"). In both studies the pupils were divided in three equivalent groups with different testing conditions. In Group 1 no special instructions were given. The pupils of Group II were explicitly instructed to make a sketch or drawing of the problem situation before computing their answer. In Group III every problem was accompanied by a correct drawing.

The major results of these studies can be summarized as follows. First, we observed an extremely strong main effect of the task variable "proportionality" in both studies: in study 1 (with the 12-13-year olds) the proportional items elicited 92% of correct responses, while only 2% of the non-proportional items was answered correctly; in study 2 (with the 15-16-year olds) the overall percentages of correct responses on the proportional and non-proportional items were, respectively, 93% and 17%. Second, as expected, the type of figure played a significant role in both studies: S-problems were the easiest and I-problems the most difficult kind of problem, but these differences occurred only in the non-proportional items. Third, we unexpectedly did not find a beneficial effect of the self-made or given drawings, neither for the test as a whole nor for the non-proportional items in particular.

While these two studies demonstrate pupils' very strong tendency to apply linear proportional reasoning in problem situations for which it is not suited, they do not allow a straightforward interpretation of it. More particularly, it could be argued that the remarkably low results obtained in these studies, were an artefact of the experimental conditions under which the data-collection took place, and that in a more favourable experimental setting the predominance of the linear model would probably be much less overwhelming. According to this argumentation, the extremely weak results on the non-proportional items and the absence of a positive effect of the self-made or given drawings were caused by the fact that the pupils had approached the test with the (implicit) expectation that it would consist of routine tasks only (as is actually typically the case in current school mathematics tests). Moreover, it could be asserted that the absence of a substantial effect of the self-made or given drawings was not surprising, taking into account the lack of useful reference points for measuring lengths and areas in these drawings (for more details about these arguments see De Bock et al., 1998).

Starting from the available results and their multiple interpretations, a new study was set up in which the administration of the proportional and non-proportional items was experimentally manipulated in two different ways to get a clearer picture of the predominance and the changeability of the linear model. More specifically, we focused on the role of the following two scaffolds on pupils' solutions of word problems involving length and area of similar plane figures: (1) a metacognitive scaffold, aimed at enhancing pupils' mindfulness while doing the test, and (2) a visual scaffold, aimed at increasing the efficacy of their actual use of drawings.

METHOD

Two-hundred-and-sixty 12-13-year olds and hundred-and-twenty-five 15-16-year olds participated in the study. In both age-groups pupils were matched in four equivalent subgroups. In all four subgroups the same paper-and-pencil test as in the two previous studies was given to the pupils, but the administration of the test was different four each group. In the M-V- group (= no metacognitive and no visual scaffold group) no special help was given. In the M+V- group (= the metacognitive scaffold group), the test was preceded by an introductory task involving the following non-proportional item about a cube:

> "A wooden cube with side 2 cm weighs 6 grams. How heavy is a wooden cube with side 4 cm?"

This item was accompanied by two alternative solution strategies - an incorrect one based on linear proportional reasoning and the correct one based on non-proportional reasoning -, and pupils were asked to select the correct one and to motivate their selection. In the M-V+ group (= the visual scaffold group), every item in the test came with an appropriate drawing of the problem situation on squared paper. Finally, in the M+V+ group (= the metacognitive and visual scaffold group) both kinds of help were combined.

HYPOTHESES

On the basis of the previous studies (De Bock et al., 1996, 1998), we first hypothesized that the predominance of the linear model would be a serious obstacle for the majority of the pupils of both age-groups. Consequently, we predicted that pupils' overall performance on the test would be very low, due to their low scores on the non-proportional items.

Second, based on the hypothesis that several years of secondary school mathematics will bear a positive effect on pupils' ability to resist and overcome the "linearity trap", we predicted that the 15-16-year olds would perform better on the test in general and on the non-proportional items in particular than the 12-13-year olds.

Third, for reasons which are explained in detail in De Bock et al. (1998), it was predicted that pupils' performance would be different for the distinct types of plane figures involved in the study. More specifically, the S-items were expected to be the easiest and the I-items the most difficult.

Fourth, we hypothesized that confronting pupils with a non-proportional problem and forcing them to make a deliberate choice between the incorrect (linear) and the incorrect (non-linear) solution at the onset of the test, will have a beneficial effect on the mindfulness with which they approach (the non-proportional items in) the test. Therefore, we predicted that the two groups receiving the metacognitive scaffold (= M+V- and M+V+) would perform better on (the non-proportional items in) the test than those who did not receive this scaffold (= M-V- and M-V+). Furthermore, it was hypothesized that the metacognitive support would be more effective for (non-proportional) items about squares than for those dealing with circles or irregular figures (because the geometrical figure of the introduction problem resembled most that of the S-items), and also that this scaffold would be more effective in the oldest age-group (because these pupils have greater mastery of the mathematical knowledge and skills needed for solving non-proportional items).

Fifth, we anticipated a better performance on (the non-proportional items in) the test for the two groups who received the visual scaffold (= M-V+ and M+V+) than for those who did not get it (= M-V- and M+V-). This prediction was based on the hypothesis that the availability of these drawings on squared paper would be of considerable help to pupils who had difficulties in modelling and solving the (non-proportional) items, by helping them see the relationships between the lengths and the areas of the two plane figures involved in the problem and by suggesting an informal but efficient solution method based on "paving". Moreover, we hypothesized that - just as the metacognitive scaffold - the visual scaffold would interact with the two other experimental variables, namely the nature of the figures involved in the (non-proportional) items and the age of the subjects. With respect to type of figure, it is evident that providing a drawing of the problem on squared paper is much more helpful when the (non-proportional) item deals with squares than when it involves circles or irregular figures. Therefore, we predicted that the drawing effect would be greater for (non-proportional) S-items than for (non-proportional) C- and I-problems. Moreover, we anticipated a greater facilitating effect of the visual scaffold in the oldest age-group because 15-16-year old pupils are already more experienced in effectively applying heuristic methods (including the use of drawings and other problem visualisations).

Sixth, we hypothesized a cumulative effect of the metacognitive and the visual scaffold. The rationale behind this hypothesis is that pupils receiving both scaffolds do not only receive a strong warning that not all items in the test are standard proportional problems; moreover, they are armed with an extra tool for modelling and solving these difficult and unfamiliar problems in an intuitive, context-bound, graphical way requiring little or no sophisticated formal-mathematical knowledge. Therefore, we predicted that the best performance on (the non-proportional items in) the test would come from the group receiving both the warning and the drawings (M+V+).

RESULTS

Table 1 gives an overview of the percentage of correct responses for the four groups of 12-13- and 15-16-year olds on the proportional and the non-proportional items about squares (S), circles (C) and irregular figures (I) in the test.

	12-13-year olds						15-16-year olds					
	Proportional items			Non-proportional items			Proportional items			Non-proportional items		
	S	C	I	S	C	I	S	C	I	S	C	I
M-V-	92	98	95	8	5	1	90	94	98	32	15	2
M+V-	92	100	95	9	3	0	74	80	97	44	21	12
M-V+	95	98	90	19	8	2	82	87	97	29	18	3
M+V+	84	90	91	24	10	0	87	78	97	60	23	8

Table 1 *Overview of the results*

As in the two previous studies, the results provided a very strong confirmation of the first hypothesis. Indeed, an analysis of variance revealed an extremely strong main effect of the task variable "proportionality" ($F(1,377) = 3293.95$; $p < .01$): for the two age-groups and the four different groups together: the percentage of correct responses for all proportional and for all non-proportional items was 91% and 15%, respectively.

The second hypothesis concerning the impact of the age factor was also confirmed: the 15-16-year olds performed better on the test in general than the 12-13-year olds ($F(1,377) = 20.77$; $p < .01$); percentages of correct answers were, respectively, 55% and 50%. Furthermore, an interaction effect between the variables "age" and "proportionality" was found ($F(1,377) = 55.83$; $p < .01$): the 15-16-year olds performed better than the 12-13-year olds on the non-proportional items (22% and 7% correct responses, respectively), but these better scores on the proportional items were accompanied by lower scores on the proportional items (88% and 93% correct responses, respectively).

In line with the third hypothesis, the type of "figure" involved in the problem had a significant main influence on pupils' performance ($F(2,754) = 38.21$; $p < .01$), and the scores were in the expected direction (58%, 52%, and 49% correct responses on the S-, C-, and I-items, respectively). Also, an interaction effect was found between "figure" and "proportionality" ($F(2,754) = 107.43$; $p < .01$). While the percentages of correct responses on the non-proportional items were in the expected direction (28%, 13%, and 4% for the S-, C-, and I-items, respectively), the percentages of correct responses on the proportional items were in the opposite direction (87%, 91%, and 95% for the S-, C-, and I-items). These results indicate that the above-mentioned main effect for type of figure was completely due to the observed differences within the non-proportional items.

Fourth, we did not find an overall beneficial effect of the metacognitive scaffold on pupils' performance on the test as a whole - the overall performance of the pupils from the M+V- and the M+V+ groups was almost exactly the same as that of the pupils from the M-V- and the M-V+ groups (53% and 52%, respectively). But the analysis of variance revealed the expected "warning × proportionality" interaction effect ($F(1,377) = 15.50$; $p < .01$). The warning did have a positive effect on the scores of the non-proportional items, but the size of this effect was rather small: the percentage of correct responses in the groups with and without the metacognitive scaffold were 18% and 12%, respectively. The better performance on the non-proportional items in the M+ groups was parallelled with a weaker score on the proportional items (i.e., 89% correct answers versus 93% for the M- groups). Furthermore, the hypotheses concerning the differential impact of the metacognitive scaffold on the distinct types of non-proportional problems was also confirmed. The analysis of variance revealed a significant "warning × proportionality × figure" interaction effect ($F(2,754) = 5.45$; $p < .01$). The percentages of correct answers on the non-proportional S-, C-, and I-items were 34%, 14%, and 5%, respectively, in the groups with warning and 22%, 11%, and 2%, respectively, in the groups without warning, but only the difference for the S-items was significant ($p < .01$). So, the warning did matter for the S-items, but this warning effect did not transfer from the S- to the C- and I-items. Finally, the anticipated greater effect of the warning in the oldest age-group, was confirmed too. The analysis of variance revealed a significant "warning × proportionality × age" interaction effect ($F(1,377) = 7.35$; $p < .01$). The percentage of correct responses on the non-proportional items in the groups with and without warning were 8% and 7%, respectively, for the 12-13-year olds, and 28% and 16%, respectively, for the 15-16-year olds. In other words, only the 15-16-year olds took advantage from the metacognitive scaffold.

Fifth, we also did not find an overall effect of the visual scaffold: the test performance of the pupils who received the drawings on squared paper was not significantly better than that of those who did not receive these drawings; percentages of correct answers were, respectively, 53% and 52%. But the analysis of variance revealed a significant "drawing × proportionality" interaction effect

($F(1,377) = 6.96$; $p < .01$): the visual scaffold did have a small but significant positive effect on the scores for the non-proportional items (the percentage of correct responses in the two groups with and without drawings were 17% and 13%, respectively), but, once again, the better performance on the non-proportional items in the M-V+ and the M+V+ groups was accompanied by a weaker score on the proportional items (90% correct answers, versus 92% for the two no-drawing groups). Finally, the results did not support the hypotheses that the drawings would have the greatest effect on the performance on the non-proportional S-items and in the oldest age-group. The analysis of variance revealed neither a significant "drawing × proportionality × figure" interaction effect, nor a significant "drawing × proportionality × age" interaction effect, implying that the above-mentioned "drawing × proportionality" interaction effect manifested itself equally in the distinct problem types (i.e., S-, C-, and I-problems) and for the two age-groups of pupils (i.e., the 12-13- and 15-16-year olds).

Sixth, the results did not support the hypothesis concerning a cumulative effect of the metacognitive and the visual scaffold. There was neither a "warning × drawing" effect nor a "warning × drawing × proportionality" interaction effect, which means that both scaffolds did not add up.

CONCLUSION

In two earlier studies (De Bock et al., 1996, 1998) the strength and omnipresence of the linear model was demonstrated with respect to problems involving length and area of similar plane figures in 12-16-year old pupils. In the present study we investigated the effect of the following two context variables on the (inappropriate) use of the linear model by pupils of the same age level who were confronted with the same problem set: (1) the provision of a metacognitive scaffold in the form of an introductory task which was aimed at enhancing the mindfulness with which the pupils would make the test, and (2) the provision of visual scaffolds in the form of a drawing of each problem made on squared paper, which provided pupils useful reference points for measuring lengths and areas and - thereby - helped them in different ways to model and solve the problems properly.

The major research question was to what extent these two manipulations of the "experimental setting" would lead to a considerable decrease in the number of incorrect answers based on inappropriate linear modelling on the non-proportional items in the test. The greater the improvement in pupils' test scores (on the non-proportional items) as a result of these experimental manipulations, the more evidence we would have that the "alarming results" obtained in our previous studies (De Bock et al., 1996, 1998) were not as frightening as we might initially have thought. On the other hand, if we found that the warning and the drawings would have only a marginal impact on pupils' scores (on the non-proportional items), this would yield further, and even more convincing support for the strength, the omnipresence and the obstinacy of the "illusion of linearity" among secondary school pupils.

The study yielded significant effects in the expected direction of both kinds of scaffolds on pupils' performance on the non-proportional items. As a drawback of these better results on the non-proportional items in the scaffolded conditions, the pupils' results on the proportional items decreased. Apparently, the scaffolds made it - at least for some pupils - easier to discover and resolve the non-proportional nature of a problem, but as a result they sometimes began to question the correctness of the linear model for problem situations in which that model *was* appropriate. However, the most important result of the present study is that the positive effects of the two scaffolds on pupils' solutions of the non-proportional items were (very) small and restricted to certain kinds of problems and age-groups. In this respect, we remind that **the combination of both scaffolds** still did yield 40% incorrect answers on **the easiest problem type** (S-problems) in **the oldest age-group** (15-16-year olds) (see Table 1)!

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TO TEACH DEFINITIONS IN GEOMETRY OR TEACH TO DEFINE?

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This paper argues from a theoretical standpoint that students should be actively engaged in the defining of geometric concepts like the quadrilaterals, and presents some data relating to a teaching experiment aimed at developing students' ability to define.

Introduction

Already early in this century the German mathematician Felix Klein (1924) came out strongly against the practice of presenting mathematical topics as completed axiomatic-deductive systems, and instead argued for the use of the so-called "*bio-genetic*" principle in teaching. The genetic approach has also been advocated by Wittmann (1973), Polya (1981), Freudenthal (1973) and many others. Essentially, the genetic approach departs from the standpoint that the learner should either retrace (at least in part) the path followed by the original discoverers or inventors, or to retrace a path by which it could have been discovered or invented. In other words, learners should be exposed to or engaged with the typical mathematical processes by which new content in mathematics is discovered, invented and organized. Human (1978:20) calls it the "*reconstructive*" approach and contrasts it as follows with the so-called "*direct axiomatic-deductive*" approach:

> "*With this term we want to indicate that content is not directly introduced to pupils (as finished products of mathematical activity), but that the content is newly reconstructed during teaching in a typical mathematical manner by the teacher and/or the pupils.*" (freely translated from Afrikaans)

The didactical motivation for the reconstructive approach includes, among others, the following elements, namely, that its implementation highlights the *meaning* (actuality) of the content, and that it allows students to *actively participate* in the construction and the development of the content. With different content (definitions, axiom systems, propositions, proofs, algorithms, etc.) one can of course distinguish different mathematical processes by which that content can be constructed (eg. defining, axiomatizing, conjecturing, proving, algorithmatizing, etc.). A genetic or reconstructive approach is therefore characterized by not presenting content as a finished (prefabricated) product, but rather to focus on the genuine mathematical processes by which the content can be developed or reconstructed. Note however that a reconstructive approach does not necessarily imply learning by discovery for it may just be a reconstructive explanation by the teacher or the textbook.

Defining

The direct teaching of geometry definitions with no emphasis on the underlying process of defining has often been criticised by mathematicians and mathematics educators alike. For example, already in 1908 Benchara Blandford wrote (quoted in Griffiths & Howson, 1974: 216-217):

> *"To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity stimulated by appropriate questions, is both interesting and highly educational."*

The well-known mathematician Hans Freudenthal (1973:417-418) also strongly criticized the traditional practice of the direct provision of geometry definitions claiming that most definitions are not preconceived, but the finishing touch of the organizing activity, and that the child should not be denied this privilege. Ohtani (1996:81) has argued that the traditional practice of simply telling definitions to students is a method of moral persuasion with several social functions, amongst which are: to justify the teacher's control over the students; to attain a degree of uniformity; to avoid having to deal with students' ideas; and to circumvent problematic interactions with students. Vinner (1991) and many others have presented arguments and empirical data that just knowing the definition of a concept does not at all guarantee understanding of the concept. For example, although a student may have been taught, and be able to recite, the standard definition of a parallelogram as a quadrilateral with opposite sides parallel, the student may still not consider rectangles, squares and rhombi as parallelograms, since the students' concept image of a parallelogram is one in which not all angles or sides are allowed to be equal.

Linchevsky, Vinner & Karsenty (1992) have further reported that many student teachers do not even understand that definitions in geometry have to be economical (contain no superfluous information) and that they are arbitrary (in the sense, that several alternative definitions may exist). It is plausible to conjecture that this is probably due to their past school experiences where definitions were probably supplied directly to them. It would appear that in order to increase students' understanding of geometric definitions, and of the concepts to which they relate, it is essential to engage them at some stage in the process of defining of geometric concepts. Due to the inherent complexity of the process of defining, it would also appear to be unreasonable to expect students to immediately come up with formal definitions on their own, unless they have been guided in a didactic fashion through some examples of the process of defining which they can later use as models for their own attempts. Furthermore, the construction of definitions (defining) is a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalizing, specializing, proving, etc., and it is therefore strange that it has been neglected in most mathematics teaching. In mathematics we can distinguish between two different types of defining of concepts, namely, **descriptive** (*a posteriori*) and **constructive** (*a priori*) defining (e.g. compare Krygowska, 1971; Human, 1978:164-165; De Villiers, 1986;1994).

Descriptive defining

"... the describing definition ... outlines a known object by singling out a few characteristic properties". - Hans Freudenthal (1973 : 458)

With the descriptive (*a posteriori*) defining of a concept is meant here that the concept and its properties have already been known for some time and is defined only afterwards. *A posteriori* defining is usually accomplished by selecting an appropriate subset of the total set of properties of the concept from which all the other properties can be deduced. This subset then serves as the definition and the other remaining properties are then logically derived from it as theorems.

Constructive defining

"... the algorithmically constructive and creative definition ... models new objects out of familiar ones" - Hans Freudenthal (1973 : 458).

Constructive *(a priori)* defining takes places when a given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process. In other words, a new concept is defined *"into being"*, the further properties of which can then be experimentally or logically explored. Whereas the main purpose or function of *a posteriori* defining is that of the ***systematization*** of existing knowledge, the main function of *a priori* defining is the production of ***new*** knowledge. We shall further on mainly focus on a discussion of the teaching and learning of the process of descriptive defining.

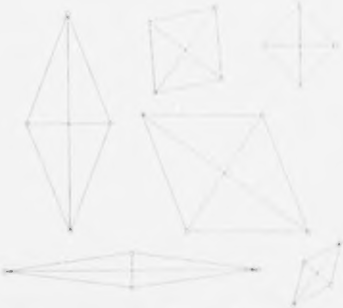
The USEME experiment

From the Van Hiele theory, it is clear that understanding of formal definitions can only develop at Level 3, since that is where students start noticing the inter-relationships between the properties of a figure. Is it possible to devise teaching strategies for the learning of the process of defining at Van Hiele Level 3? This in fact was the focus of the University of Stellenbosch Experiment with Mathematics Education (USEME) conducted with a control group in 1977 and an experimental group in 1978 (see Human & Nel et al, 1989a). The experiment was aimed at the Grade 10 (Std 8) level and involved 19 schools in the Cape Province. Whereas the traditional approach focusses overridingly on developing the ability of making deductive proofs (especially for riders), the experimental approach was (among others) aimed mainly at:

- letting students realize: (1) that different, alternative definitions for the same concept are possible; (2) that definitions may be uneconomical or economical; (3) that some economical definitions lead to shorter, easier proofs of properties
- developing students' ability to construct formal, economical definitions for geometrical concepts

The following is an example of one of the first exercises in (descriptive) defining used in the experimental approach (see Human & Nel et al, 1989b:21). Note that although these students had already come across the concept "*rhombus*", they had not been given any definition in earlier classes.

EXERCISE



1(a) Make a list of all the common properties of the figures above. Look at the angles, sides and diagonals and measure if necessary.

(b) What are these types of quadrilaterals called?

(c) How would you explain in words, *without making a sketch*, what these quadrilaterals are to someone not yet acquainted with them?

The spontaneous tendency of almost all the students in (c) was to make a list of all the properties discovered and listed in (a); thus giving a correct, but uneconomical description (definition) of the rhombi (thus suggesting Van Hiele Level 2 understanding). This led to the next two exercises which were intended to lead them to shorten their descriptions (definitions) by considering leaving out some properties.

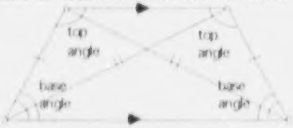
Typically the students then came up with different shorter versions, some of which were *incomplete* (particularly if they're encouraged to make them as short as possible by promising a prize!), for example: "*A rhombus is a quadrilateral with perpendicular diagonals*". This provided opportunity to provide a counter-example and a discussion of the need to contain enough (sufficient) information in one's descriptions (definitions) to ensure that somebody else knows exactly what figure one is talking about. Also note at this stage that they were not expected to **logically** check their definitions, but expected to check whether the conditions contained in their definitions provided sufficient information for the accurate **construction** of a rhombus.

Psychologically, constructions like these are extremely important for the transition from Van Hiele Level 2 to Level 3, since it helps to develop an understanding of the logical structure of "*if-then*" statements (compare Smith, 1940). For example, students learned to distinguish clearly between the relationships they **put** into a figure (the premisse) and the relationships which **resulted** without any action on their part (the conclusion).

The students were then led into a deductive phase where starting from one definition they had to logically check whether all the other properties could be derived from it (as theorems). The same exercises were then repeated for the parallelograms. Eventually, it was explained to students that it would be confusing if everyone used different definitions for the rhombi and parallelograms, and it was agreed to henceforth use one definition only for each concept.

In order to evaluate whether students had developed some ability to formally define geometric concepts themselves, the following were some of the questions given afterwards to the experimental, as well as the control group. The first question was of a known concept that both groups had already treated in class (the control group in a direct way & the experimental group in a reconstructive way). So essentially they just needed to recall a definition done in class. This question therefore served only as a base line against which to judge their ability to define in the next question which was of a completely new concept that had been not treated at all in any of the groups.

1. Give a definition of the parallelograms.
2. Quadrilaterals which look like the one below is called a *regular trapezium*.



The regular trapeziums have among others the following properties:

(1) One pair of opposite sides parallel, but not equal.
(2) Diagonals are equal.
(3) Base angles are equal (see figure).
(4) Top angles are equal (see figure).
(5) A top angle and base angle are togther equal to 180°.
(6) One pair of opposite sides are equal, but not parallel.

Answer the following questions:

(a) Provide a definition (as short as possible) of the regular trapeziums.
(b) Prove that the properties of regular trapeziums not mentioned in your definition, indeed logically follow from your definition.

Table 1 gives the results that were obtained. Note that both groups had the same teachers and that they were statistically comparable in terms of IQ, language ability, etc. It is immediately noticeable that the experimental group gave higher percentages of correct, economical definitions in both cases.

The experimental group also gave fewer correct, uneconomical definitions in both cases. This improvement in terms of economy of definition for the experimental group, however, appeared to be at a slight cost in relation to Question 1, in the sense that there was a slightly higher number of faulty definitions which contained insufficient properties. It is possible that this increase was due to uncritical attempts at producing economical definitions. This indicates a possible risk of the experimental approach. What was perhaps extremely surprising was that both the control and experimental groups performed better in defining the unknown concept than the known concept. A possible explanation could be that in Question 2, the act of constructing a definition themselves, forced them to more carefully consider the underlying logical relationships, than to just uncritically try and recall a previously learnt definition in Question 1.

		Question 1	Question 2
Correct economical	Control	25%	44%
	Experimental	54%	58%
Faulty	Control	22%	8%
	Experimental	26%	4%
Correct uneconomical	Control	51%	47%
	Experimental	19%	39%
None	Control	2%	0%
	Experimental	0%	0%

Table 1

Further discussion

From the constructivist assumption that meaningful knowledge needs to be actively (re)-constructed by the learner, it also follows that students should be engaged in the activity of defining and allowed to choose their own definitions at each Van Hiele level. This implies allowing the following kinds of meaningful definitions at each Van Hiele level (compare Burger & Shaughnessy, 1986):

Van Hiele 1: *Visual* definitions, eg. a rectangle is a quad with all angles 90° and two long and two short sides.

Van Hiele 2: *Uneconomical* definitions, eg. a rectangle is a quadrilateral with opposite sides parallel and equal, all angles 90°, equal diagonals, half-turn-symmetry, two axes of symmetry through opposite sides, two long and two short sides, etc.

Van Hiele 3: *Correct, economical* definitions, eg. a rectangle is a quadrilateral with an axis of symmetry through each pair of opposite sides.

The first two examples show that students' definitions at these levels would tend to be *partitional*, in other words, they would not allow the inclusion of the squares among the rectangles (by explicitly stating two long and two short sides). In contrast, according to the Van Hiele theory, definitions at Level 3 are typically *hierarchical*, which means they allow for the inclusion of the squares among the rectangles, and would not be understood by students at lower levels. However, research reported in

De Villiers (1994) show that many students who exhibit excellent competence in logical reasoning at Level 3, if given the opportunity, still prefer to define quadrilaterals in *partitions*. (In other words, they would for example define a parallelogram as a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal).

For this reason, students should not simply be supplied with ready-made definitions for the quadrilaterals, but allowed to formulate their own definitions irrespective of whether they are partitional or hierarchical. By then discussing and comparing in class the relative advantages and disadvantages of these two different ways of classifying and defining quadrilaterals (both of which are mathematically correct), students may be led to realize that there are certain advantages in accepting a hierarchical classification (compare De Villiers, 1994). For example, if students are asked to compare the following two definitions for the parallellograms, they immediately realize that the former is much more **economical** than the latter:

hierarchical:	A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
partitional:	A parallelogram is a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal.

Clearly in general, partitional definitions are longer since they have to include additional properties to ensure the exclusion of special cases. Another advantage of a hierarchical definition for a concept is that all theorems proved for that concept then automatically apply to its special cases. For example, if we prove that the diagonals of a parallelogram bisect each other, we can immediately conclude that it is also true for rectangles, rhombi and squares. If however, we classified and defined them partitionally, we would have to prove separately in each case, for parallelograms, rectangles, rhombi and squares, that their diagonals bisect each other. Clearly this is very uneconomical. It seems clear that unless the role and function of a hierarchical classification is meaningfully discussed in class, many students will have difficulty in understanding why their own partitional definitions are not used.

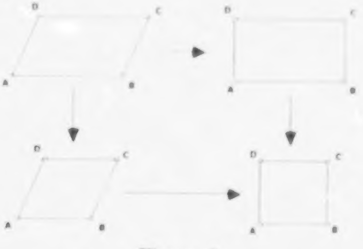


Figure 1

On the other hand, the dynamic nature of geometric figures constructed in *Sketchpad* or *Cabri* may also make the acceptance of a hierachical classification of the quadrilaterals far easier. For example, if students construct a quadrilateral with opposite sides parallel, then they will notice that they could easily drag it into the shape of a rectangle, rhombus or square as shown in Figure 1. (Recently in a session on *Sketchpad* with my 8-year old son, he had no difficulty dragging a parallelogram into the shape of a square and a rectangle, and then accepting that they were special cases). In fact, it seems quite possible that with dynamic software, students would be able to accept and understand this even at Van Hiele Level 1 (Visualization), but further research into this particular area is needed.

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STUDENT THINKING ABOUT MODELS OF GROWTH AND DECAY

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In this classroom-based research study, the thinking of two groups of precalculus students about exponential models of growth and decay. Using a multi-stage approach to model development, a curriculum unit was designed to elicit students' creation of a model that can be used to describe, explain and predict the behavior of an experienced, probabilistic system. This study thus brings together the research strand of model building with students' understanding of exponential functions and probabilistic data. The evidence presented suggests that the model eliciting activity did support students' construction and successive refinement of a model of exponential growth that became increasingly generalized and abstract.

Introduction

The mathematics of change and variation covers a broad spectrum of important mathematical ideas for secondary students, including the understanding of functions as co-varying quantities. The mathematics of uncertainty and chance likewise provides a setting for interpreting and understanding a wide range of common experiences. In this paper, I bring together these two themes in a research study on student strategies in investigating a probabilistic exponential growth and decay problem situation through a multi-stage model building approach. The first stage of model development begins by confronting the students with the need to create, in the first place, a model that can be used to describe, explain, manipulate, and predict the behavior of an experienced system. The interpretation of the problem situation is an essential part of the task. Students need to make and defend judgments about the strengths and weaknesses of alternative conjectures, assumptions, descriptions, representations and explanations. Exponential functions provide a rich mathematical site for examining students' strategies for constructing such useful and meaningful models and for gaining a better understanding of students' thinking about these functions as it is revealed by their strategies. Relatively little research has been done on students' thinking about the multiplicative structures that underlie exponential functions (Confrey and Smith, 1994) and the role of probabilistic events in relationship to these functions.

Theoretical Framework

A model building approach to learning mathematics suggests that an important goal of such learning is for students to be able to construct mathematically significant systems that can be used to describe, explain, manipulate, and predict a wide range of experiences. Most school problems that are posed to students do not involve the students in creating, modifying or extending systems of representations of meaningful problem situations. In solving typical textbook "word problems," students generally try to make meaning out of questions that are often simply a thin layer of words disguising an already carefully quantified situation. The solution process is an exercise in mapping the problem information onto an invariant model using symbolic notation in such a way that an answer can be produced. This activity rarely involves an explicit examination of the underlying mathematical model; seldom is the underlying model transformed, modified, extended or refined; rarely are students involved in the creation of such models. Model building, on the other hand, explicitly focuses on the activities of creating symbolic, graphical, and numeric representations and descriptions of situations that are meaningful to the learner; once created, such models can be explored, refined, extended and applied to other contexts.

Modifying, generalizing, sharing and re-using models are central activities for students to engage in as they learn significant mathematics. Model building is not seen as steps in finding a solution to a given problem but rather as developing a tool that a learner can use and re-use to find solutions in a range of contexts that are structurally isomorphic (Bransford, Zech, Schwartz & The Cognition and Technology Group at Vanderbilt University, 1996; Doerr, 1997). In this research study, model building is seen as a multi-stage, multi-cycle activity. Within each stage of modeling activity, students engage in multiple cycles of interpretations, descriptions, conjectures, explanations and justifications that are iteratively refined and re-constructed by the learner. This view of student's conceptual development through modeling is shaped by earlier research that posits a non-linear, cyclic approach to model building (Doerr, 1996).

The modeling process begins with the model *elicitation* stage, which confronts students with the need to develop a model to describe, explain and predict the behavior of an experienced system. Models that students have constructed can then be *explored* for their own sake so as to generalize their range of applicability, to extend their power and utility, and to make explicit the underlying patterns, regularities and structures. The stage of model exploration provides students with the opportunity to develop powerful structural metaphors to make sense of their world of experience. Models that have been explored, and possibly refined, can then be applied to new problem-solving situations that could not have been dealt with adequately without the newly constructed model. This stage of model *application* engages students in seeking new situations that can be described or explained using the model that they have constructed. A model becomes a re-usable and shareable tool to be applied by learners within new contexts.

The role of multiplicative structures, as described by Confrey and Smith (1994), provides a theoretical basis for examining student understandings of exponential functions. These researchers argue that "splitting" is an equally strong approach to understanding multiplication as is repeated addition. The multiplicative world of the exponential function has its origins in the actions of splitting, which is a distinctly different operational view of multiplication than is repeated addition. This would suggest that seeing the constancy of successive ratios should be as essential to an understanding of exponential functions as is the constancy of first order differences in linear functions. A particular instance of "splitting" would be the doubling of bacteria or the halving of radioactive decay or, in this study, increasing or decreasing the number of elements in a set by applying a 50-50 chance to each element for adding to or removing from the set. In this study, I examine how students come to understand and interpret the exponential function in a probabilistic problem situation.

Description of the Study

The instructional approach to the unit is based on the notion of providing a model eliciting activity to motivate and guide the students' inquiry and to confront them with the need to develop a model to describe and explain a problem situation. The model eliciting activity (described in more detail below) was designed to provide the students with a problem situation that they can readily understand through their own first hand experience and to provide a significantly rich mathematical site that will lend itself to refinements, extensions, and powerful generalizations (Lesh, Hoover & Kelly, 1993). The overall curricular unit is designed to provide activities that support (1) the generation of a model in the first place, (2) the exploration and possible refinement of the significant mathematical relationships, representations, and assumptions, and (3) the application of the model to other problem situations.

Setting

The setting for this study was two pre-calculus classrooms in a suburban upper middle class high school. The classrooms were an open, flexible environment where small group work is common and students are actively encouraged to express their own ideas and take responsibility for their own learning. This study took place in two classes with 17 and 13 students in grades 10 through 12, who had elected to take the course. All the students had their own graphing calculators; a graphing calculator overhead display unit was available at all times in the classroom and used spontaneously by both the teachers and the students. There was also a single computer and a printer in the room. The class met for three single periods of 40 minutes and two double period of 80 minutes each week. This particular unit lasted approximately 4 periods.

The class was taught by an experienced mathematics teacher, who was interested in including more technology based labs and student explorations as part of the curriculum. This was the first year with double period scheduling for pre-calculus classes. The teacher was very experienced with the graphing calculator and flexible in her approach to the time needed for instruction. Each class was divided into 5 small groups of 2 to 4 students. These groups provided a setting within which to observe the students' interactions with the problem situation and with each other.

Data Sources and Analysis

Each class session of the overall unit was video-taped, and during small group work, three selected groups were audio-taped and observed. All of the written work, including mid-unit and end-of-unit testing, done by all of the students was made available to the research team. Extensive field notes were taken by each member of the research team during the class sessions. The video-tapes of class sessions were reviewed and selected portions were transcribed for more detailed analysis. The research team regularly met with the classroom teacher for the planning of the unit, for revisions and modifications made during the unit, and for reflection on the students' learning.

Description of the Model Eliciting Activity

The lesson began by posing the following problem situations to the students. Each group of students had a small cup and a supply of M&M candies, a popular disk-shaped candy with an "m" on one side.

> You will start with one or more M&Ms in your cup. Shake the cup and pour the contents onto a napkin. For each M&M that has the "m" showing, add a new M&M to the cup. Put all the M&Ms back in the cup and repeat the procedure 10 more times. For each trial, record the total number of M&Ms in the cup.

The students were asked to graph their data and to find an equation that could be used to predict the total number of M&Ms for any given trial. They were asked for specific predictions for certain trials and to find how many trials would be needed to have 300 M&Ms. Most importantly, the students were asked to explain the meaning of the constants and the variables in their equations. Earlier in the instructional sequence, the students had investigated similar situations of exponential growth (e.g. the doubling of bacteria, population growth, and compound interest), but this particular problem situation was intended to extend those notions of growth to probabilistic events.

The second problem situation had the students investigate a decay situation: they were to start the experiment with a full cup of M&Ms and at each trial to remove all M&Ms that had the "m" showing. Again, they were asked to create graphs and equations, make predictions, and explain the meaning of the constants and variables in their equation. This problem situation was the first introduction to the idea of exponential decay and was intended to become the central metaphor for the students' developing understanding of the mathematical ideas underlying exponential decay. The experience of the physical phenomena was intended to ground the students' thinking in an experience that was familiar and readily understandable.

A third problem situation was designed to provide an exploration of the growth and decay models elicited in the first two activities. The students were asked to consider a cupful of hypothetical four-sided M&Ms, with each M&M having exactly one side with an "m" on it. This problem was intended to engage the students in extending, refining, abstracting and generalizing the models that they built for the first two situations.

Results

The students readily engaged with the problem situation in their small group and the findings of the groups led to a rich and lively whole class discussion. In this

13/23?" This is (from their table) the ratio of the increase in the number of M&Ms to the number of M&Ms in the cup. The teacher who has joined them asked: "what's changing?" S1 replied "the number of M&Ms with m's up. The number of M&Ms we added?" The teacher focused their attention on the kinds of quantities they are comparing as she questions: "What are you comparing?" "What is the rate of change?" and "What per what?" S1 replied slowly: "So 4 M&Ms per (pause) per 9 M&Ms?" Clearly S1 was still thinking about the relationship between the total number of M&Ms and the number of M&Ms with m's showing. The teacher asked pointedly: "what are your variables?" and S1 clarified that x= number of M&Ms in the cup and y = the # of m's showing.

The teacher left and the students sat quietly, thinking. S2 returned again to her premise "I'm thinking it might be exponential." S3 objected "but it looks more linear." Then S2 really makes explicit that one of the variables should be the trial number: "But the trials are the numbers" and "we need trials to be x" and then she draws on an analogy to other unspecified problem situations: "in all the other ones it was years". S3 graphed trials as the x variable and showed it to S2 who says "That's good! What's the equation?" S3 gave the regression equation "Write y = 1.039 (1.46)^x." S1 commented that they need to "figure out what these things mean." This question was posed in the original problem, but they did not engage with this question, perhaps in part due to a sense that time is running out. They collected the data for the second problem, but without much discussion or any conjectures. The model they have is one which has an equation from regression analysis, a table and a graph, but certain things have not yet been integrated into their growth model, namely the meaning of the constants in the regression equation, the randomness of the data, the halving that occurred in the experiment, and how to account for the linearity of the alternative model. This latter element becomes one of the unexplored aspects of their model from a mathematical point of view. It is a piece of their overall model that has been filtered or selected out. The decay model is barely developed nor integrated into the growth model; the generalizations of the hypothetical experiment were not addressed in their small group setting.

A second group of students moved somewhat differently through the activities. Like the first group, this group of two students began by collecting their data and constructing a table (see Figure 2). They began with 5 M&Ms in the cup, initially. They did not give headings to their two columns of data and, unlike the first group, began counting their trials at zero rather than one. The students looked quickly at the magnitude of the first order differences and then immediately graphed the number of M&Ms versus trial number. They easily adjusted the window to get a view of the data. S4 suggested that the data is "something of per cent increase, something increasing." S5 made a connection to a previous problem situation, saying it's "like compound interest." S4 then claimed the data is "like 1.5^x only not curvy enough." S5 identified the initial value as critical and suggested "start with 5 times something." S4 reaffirmed S5's suggestion and they seem satisfied with their equation. S5 hinted section, I will compare and contrast the strategies taken by two different groups, one in each of the classes. This analysis focuses on the multiple modeling cycles which occurred within each group, the extent to which ideas about probability emerged and influenced student thinking, and student thinking about exponential functions.

The first group of students began by collecting and recording the data shown in Figure 1. The teacher had given each group within the class a different starting number of M&Ms in their cup. This particular group started out with two M&Ms. The group decided to record both the total number of M&Ms in the cup and the change in the number of M&Ms for each trial. While their initial language referred to this number as the change, as they collected the data they recorded and referred to this as the number of m's showing. There was no discussion or conjecturing as they were collecting the data, but rather a focus on counting and recording.

trial	# of M&Ms in cup	# of m's showing
1	2	0
2	2	1
3	3	1
4	4	3
5	7	3
6	10	4
7	14	9
8	23	13
9	36	12
10	48	24
11	72	33

Figure 1.

When the table was completed, one student, S1, proposed the first strategy to find a relationship between the two quantities by suggesting they graph "how many M&Ms are in there.. horizontally.. versus how many have m's showing. Ignore the first column." Another student, S2 offered a mild challenge to this suggestion and said "let's do like two graphs," although it was not yet clear which two graphs she was thinking about. In the meantime, the third student, S3, entered the 2nd and 3rd columns of data into her calculator and suggested that the data "is gonna be linear." But S2 opposed the linear conjecture and said "no, it's going to be like the doubling one" (referring to the bacteria problem done earlier in the unit).

As the students continued to negotiate the competing claims of linear and exponential growth, S3 introduced a quantification of the rate at which the total is changing. S3 said "you are taking half each time - listen" and S1 agreed "half the M&Ms [are] showing up [with an m] each time." S2 used this evidence to re-affirm and re-assert her claim that "I think it's exponential." It would appear that the increase by halving is isomorphic for S2 to the increase by doubling. S3 continued to look at the linear regression, and she finally exclaimed: "No, I don't understand how to figure this. I don't know how to figure this out from this random data." This is the first time that the randomness of the data has been identified as an issue. S1 asked "shouldn't the y-intercept be 2 because we started with 2?" It's clear that the somewhat linear data does not have a y-intercept of 2. It's less clear what S3 finds unsatisfactory about the linear regression equation. At this point, the group appears to be at an impasse.

Without a model in place, they moved on to the next question which is to estimate the average rate of change on the 8th trial. S1 quickly said: "Wouldn't it be at a difference between this situation and others that they have worked on: "You can't calculate because of human error. I think that's a valid equation though." S4 articulated the meaning of the constants, saying "1.5 makes a lot of sense. 1 plus 50% cause 50% of the time you get an m. Like compound interest." At this point they have created and justified a model t that is a fairly good match with the data they have collected and is analogous to similar problems they have seen earlier in the unit. They use their equation and data table to unproblematically make several predictions about the behavior of the system.

5	0
8	1
12	2
18	3
26	4
42	5
58	6
88	7
125	8
198	9
292	10

Figure 2.

These students then moved to the decay problem. S4 began by suggesting "why don't we just write out the equation" -- implying that he already had generalized, or at least had an idea how to generalize, the situation. S5 responded with a partially tongue-in-cheek "I don't want to strain my brain." They quickly collected their data, entered it into their calculators and graphed it. S4 immediately began to hypothesize "parabola, square roots, 1 over something, 1/x, not what I wanted." S5 sat and thought and hypothesized y = 130*1.5^(-x), arguing "seems like it would decrease at same rate it increased at?" S4 challenged this saying "No, I'm not sure it should be 1.5." S5 engaged in an iterative guess and check strategy: "I tried something lower than 1.5 and it was worse. 1.7 is extremely close." S4 also continued with the guess and check strategy even though he is somewhat dissatisfied with a strategy that doesn't make sense, but he knows that the "school goal" and perhaps their own goal is "to make it work." S4 did not find a regression equation, which he was quite capable of doing, but rather stayed with a sense making strategy. S5 said 1.8 looks really nice. S4 responded "I wonder. 1/2 looks best which makes even more sense. 1/2 to the x makes more sense. Normal growth...only more sense. What did everyone else get? y=130*(1/2)^x." S4 graphed this and said it "fits perfectly." S4 explained the meaning of the constants and variables in their equation, but puzzled over the (1/2)^x. She asked what it is the inverse of? S5 responded that it is the inverse of the normal growth curve. This answer is accepted unproblematically.

They turned to the hypothetical situation of the growth situation for the 4-sided M&M. S5 quickly generalized: "Say we started out with 5 of these cube M&Ms. 5(1.75)^x." S4 questioned if that's will or won't get an m and S4 changed his expression to 5(1.25)^x. S4 shifted his thinking to the physical situation and compared it to the 2-sided M&M: "Do we end up with less in the cup [compared to the 2 sided M&M] at the end? Yes, so we want it to be 1.25. We end up with less." S4 compared the 2 sided M&M growth to the 4 sided M&M growth as a piece of sense making about the situation as well as a generalization. S5 focused on the time rather than the quantity (as S4 did) and she observed that "it would take longer" to arrive at the same final quantity for a 4-sided M&M. They were comfortable that this is two different ways of looking at the same thing rather than an issue that has to be resolved.

Discussion and Conclusions

Both groups of students in this study created meaningful interpretations and useful representations of the probabilistic growth situation. The first group encountered a conflict between a linear model and an exponential model to describe their data, which focused on the total quantity and the change in the quantity. This conflict was resolved by an examination of the rate of change in the total quantity, which forced the students' attention to an identification of the kind of quantities. This, in turn, led to an exponential model that was defined through a regression equation, but that equation did not provide the students with explanatory power. That is, there was no answer as to why these particular constants showed up in this particular way in this equation. At least one student in the group, however, made strong analogical arguments, drawing on the "halving" which occurred, that this would argue for an exponential model.

The second group of students quickly drew on previous knowledge about compound interest and percentage increases to develop an equation that fit their data, that made sense in terms of previous problem situations, and that had explanatory power for the situation at hand. Their interpretation of the decay problem was as the inverse of the growth problem, but by negating the exponent. They worked through an iterative refinement of this initial hypothesis to arrive at a base which again fit their data and had some explanatory power for the situation at hand. They abstracted and generalized their findings to a hypothetical experiment for the growth situation.

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ANALYSIS OF A LONG TERM CONSTRUCTION OF THE ANGLE CONCEPT IN THE FIELD OF EXPERIENCE OF SUNSHADOWS

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In this paper, I shall describe how different aspects of the angle concept emerged and evolved during a long term classroom activity involving progressive geometrical modelisation of the phenomenon of sunshadows. I shall try also to analyse how the idea of "inclination", related to mastery of the angle concept in space, matured in a four-step sequence of individual and collective activities.

1. Introduction

This report deals with a "problematique" regarding the multiplicity of the angle concept: the multiplicity of embodiments and reference situations, multiple relationships between dynamic and static aspects, etc. This has a long history in mathematics education: from Freudenthal's educational analysis (Freudenthal, 1973; 1983), up to studies related to the Logo environment (see Clements & Battista, 1992) and recent investigations concerning the embodiment of the angle in turning situations - see Mitchelmore & White, 1996.

This report focuses on the multiplicity of the angle concept as met by students in a long term teaching and learning activity in the "field of experience" (Boero & al, 1995) of sunshadows from the beginning of grade III to the end of grade IV. I have studied how this multiplicity emerged (see 4.), evolved and matured within tasks demanding a progressive geometrical modelisation of the sunshadows phenomenon. In this paper I will also consider a sequence of two individual and two collective activities related to the same problem situation (see 5.). I will try to elicit the ways by which maturation of the idea of "inclination" was reached.

2. Theoretical framework

About Concepts in General

I will mainly refer to Vergnaud's definition of concept (Vergnaud, 1990), as "reference situations", "operational invariants" and "linguistic representations". The "multiple" and "evolutive" characteristics of a concept inherent in Vergnaud's theoretical framework are particularly stressed in other elaborations which concern the idea of "concept" and belong to other domains. Deuleuze and Guatari's elaboration about concepts in philosophy, concerning their "multiplicity" and their "becoming" (Deleuze & Guattari,1991), suggests that multiplicity should be considered from an evolutive perspective. Nelson's studies (Nelson, 1978) about the genesis and evolution (from implicit to explicit, from particular to general) of concepts from a psycholinguistic point of view suggest the different levels at which concepts can be interiorized and treated by students.

About the Angle Concept

In the light of these elaborations, we can consider the multiplicity of the angle concept. In this report I will describe how different aspects[1] of the angle concept emerged and evolved in a primary school class over a period of 15 months. I will consider not only the static and the dynamic aspects, but also others which can only roughly be reduced to the static-dynamic polarity. For each aspect of the ángle concept I will consider the reference situations (all related to the field of experience of sunshadows) and the different levels of emergence and treatment (corresponding to linguistic representations and operations in Vergnaud's definition): bodily experience, verbal and graphic description of spatial relations, etc.

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[1]: "Aspect", or ....? The multiplicity of the angle concept does not consist in the splitting of the "reference situations", "operational invariants" and "linguistic representations" into *separate* subsets. For this reason I shall use the expression "aspect of the angle concept". Indeed each "aspect" may involve some situations, invariants or representations in common with other "aspects", and furthermore does not stand alone in the evolution of students' knowledge.

Inclination and Angle

In Book I of "Elements" (see Heath, 1908), Euclid defines a *plane angle* as *the inclination to one another of two lines in a plane...* (def. 8). So, inclination works as the defining concept for angle. But in Book XI, concerning stereometry, Euclid defines the *inclination of a straight line to a plane* in this way:*"Assuming a perpendicular drawn from the extremity of the straight line which is elevated above the plane to the plane, and a straight line joined from the point thus arising to the extremity of the straight line which is in the plane, [it is] the angle contained by the straight line so drawn and the straight line standing up"*. As Serres (1993) remarks, this definition (although the straight line and the plane are in generic positions!) brings a strong sense of "inclination" related to a *horizontal* plane: *"elevated above the plane"*, *"the straight line standing up"*. In this way, "inclination" (coherently with its Greek etymology) keeps a meaning of property of a line related to the bodily reference; the plane angle enters the definition as the defining concept which will be exploited for measurements, equivalences, etc. Historical and epistemological analysis suggests that the relationship between inclination (especially, inclination intended in its space and bodily meaning) and angle deserves special attention from the cognitive and educational points of view.

About Context

I shall use Boero's definition of "field of experience" (Boero & al, 1995), especially as concerns the ideas of external context and student's internal context. For a given subject (in our case, "sunshadows"), the theoretical construct offers guidelines for following the long-term development of the student's "internal context" (i.e. her/his conceptions, schematas, etc.) in relation to the "external context" (signs, concrete objects, physical constraints, etc.).

The idea of *grounding metaphors* (proposed by Lakoff & Nunez,1997), which *"allow us to ground our understanding of mathematics in familiar domains of experience"*, suggests that we should try to identify the different aspects of the angle concept within different representations (gestural, graphic, verbal...) in the observed classroom activities concerning sunshadows.

3. The class, the educational context, the available data

Last year I was offered the opportunity of a one-month visit to an Italian fourth grade class where the teachers (Ezio Scali and Nicoletta Sibona) had been carrying out, since grade I, the Genoa Group project for primary school. The aim of this project is to teach mathematics, as well as other important subjects (native language, natural sciences, history, etc.), through systematic activities concerning "fields of experience" from everyday life (see Boero et al, 1995). The sunshadows field of experience is the ground, in grades III, IV and V, for developing argumentative skills and geometry concepts.

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A sequence of four didactic situations was negotiated with the teachers. This sequence represented a fairly common classroom routine consisting of: individual production of written hypotheses on a given task; classroom comparison and discussion of student products, guided by the teacher; individual written reports about the discussion; classroom summary, usually constructed under the guidance of the teacher and finally written up in the copybooks.

In most cases (as indeed in this one), classroom summary represented the status of the knowledge that the students reached (with all possible ambiguities and hidden mistakes), and not a phase of final institutionalisation (Brousseau, 1986), something which was attained only in a few circumstances. This style of slowly evolving knowledge without "sure" and final "truth" offered me the opportunity to observe the transformation of the students' knowledge in a favourable climate where this transformation was a normal, expected event.

The data used as a grounding for the analysis of the sequence derive from direct observation of the four didactic situations, all the students' texts, and videos (and transcripts) of all classroom discussions (see Section 5). The class being one of the Genoa Group's "observation classes", a lot of information was accessible about previous activities (see Section 4): students' individual texts, students' copybooks, recordings of classroom discussions, and one video.

In this class, activities concerning sunshadows (and angles, a topic almost exclusively treated in this field of experience) started in January, 1996 (grade III) and ended in December, 1997 (grade V). I will consider only the activities performed in the period January 1996 to April 1997.

4. Students meet the angle concept: an overview

4.1. Embodied, implicit approach to the angle in space: inclination

Students experienced some aspects of the angle concept by indicating sun and its apparent movement in the sky with their arms, then discussed about their observations and drew the situation. The outline of one student's shadow on the ground was traced with chalk and (a few days later) drawn on a large sheet of paper at different hours of the day and hung to the main wall of the classroom. Photographs of the children standing in a row pointing at the sun were taken, observed and commented on. The repeated observation and discussion of the relationships between different positions of their bodies and the shadows on the ground was performed. Students repeteadly were: actors (moving their bodies, arms, eyes); objects (with their shadows) of observation and verbal and graphic representation by their schoolfellows; and represented objects (in drawings, photos, texts). This plurality of roles gave opportunity of multiple embodiments of the idea of "inclination" (as phisically experienced, as observed and represented, and as readable in different representations).

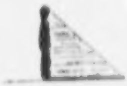
During these long term activities (more than 30 hours in grade III) the word "angle" was neither needed nor produced; on the contrary, the words *"inclinazione"* ("inclination") and *"direzione"* ("direction") were used as common Italian language terms (without specific reflection on their meaning). In this phase, we may note that, when students pointed at the sun, the word "inclination" took on a static meaning (i) of a direction relative to the earth's plane in tridimensional space, but also a dynamical meaning (i') linked with the movement of the sun when they indicated the movement of the sun with their arm and wrote that *"the arm indicating the movement of sun in the sky during the morning moves upwards and rightwards "*. The intersection of the line given by the arm with its projection on the plane of the ground was not visible nor important in this phase.

4.2. Plane Angles and their Reference Situations

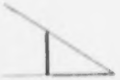
I have selected four activities carried out in grade IV from October to March, which covered about 30 hours of classroom work on sunshadows.

a) - Gradual construction of the "shadow schema"

Initially, students drew themselves standing in the sun, their shadow and the shadow space. This activity demanded an important bodily activity: moving the hands in different positions near and away from the body in order to discover the existence of the shadow space and find its border.



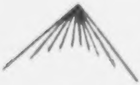
This work gave birth to a first schema which gradually became a static geometric modelisation of the spatial dynamic relationship between the inclination of sunrays and the length of shadows, establishing an initial link between the angle in space and the angle represented in the plane of the sheet of paper.



At the end of different student activities involving the use of this schema, the teacher (see Scali, 1997) introduced the shadow schema as a tool for interpreting some aspects of the sunshadow phenomenon.

During the activities concerning the first and second schemas, the drawn angles between the sunray and the body and between the sunray and the ground were probably seen as an indication of the borders of the shadow space (ii): a limited surface) and (iii) the intersection of two directions: the direction of sun rays and the section of the earth's plane. For most of the students, the width of the shadow space (ii') (as the *"distance between its border and the body"*) remained for a time the (ambiguous) characterisation of what was to gradually become the "inclination of sunrays". In this phase, neither angle was the object of reflective work, nor was it named; it remained an implicit tool in the graphic representation of the relationship between the body, the shadow space and the sun.The *complexity* of the observed (and bodily experienced) situation and of the activity of graphical representation created interesting ambiguities that were gradually cleared up through *argumentation* (see d) and 5.)

b) Shadow fans



Later on, the students observed and traced out on a big sheet of paper the "shadow fan" of the shadows cast by a fixed nail, which were measured every hour throughout the day.

This was repeated once a month, and the students compared the fans, analyzing the changes during the year.They also produced and compared shadow fans of nails of different lengths. This activity implicitely concerned the angle seen as a description of the constant rotational displacement of a direction independent of the length of the segments representing it, corresponding to the effect of the plane rotation (iv)of the direction of a segment (performed in an hour); thus, we may recognize here the element characterising a geometrical transformation (rotation)(iv'). In the shadow fans, there still remained a representation of angles with the meaning of the gap between two directions in the plane(ii'')

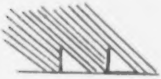
c) - Scale reduction of the shadow fan

The students were set the task of reproducing (with necessary reduction!) the large hourly shadow fans of 8-10 cm nails in their copy books: this open problem situation was tackled individually, then a discussion followed (with short final summary).In their individual solutions, many students tried to preserve particular aspects of the original shadow fan as a representation of the phenomenon of sunshadows during the day: some of them focused on the invariability of angles through reduction(v) in order to preserve the whole shape of the shadow fan. In this way, they experienced (as a theorem in action, Vergnaud) angle (v') as a characteristic of a geometric transformation, homothety. Other students tried to preserve the width of the rotation as a priority in order to reproduce the same rotation effect between the same moments even though the segments are shorter, thus preserving the angle as characterising another transformation, the rotation (iv').

Discussion concerned these two strategies (based on the static and dynamic meanings of the angle - see Freudenthal, 1983). During the discussion, the word "angle" was spontaneously used in an informal way by some students. Subsequently, the teacher encouraged more and more precise usage, but without any definition. In the final summary of the discussion the students reported the two reasons they had found for *"preserving the angles"*.

d) Displacing the nails

The sign originally proposed by the teacher ("shadow schema") did not contain the sun (indeed, in a frontal position - i.e., when the observer looks at the nail perpendicularly to the shadow - the sun cannot be seen!). But immediately after, the students started to add the sun as a reference for the origin of sun rays. In order to begin challenging the conventionality of the representation of sun within the shadow schema, the teacher asked the class to hypothesize the effect that displacing an object from one courtyard to another would have on the length of its shadow at a given moment. Experiments followed.



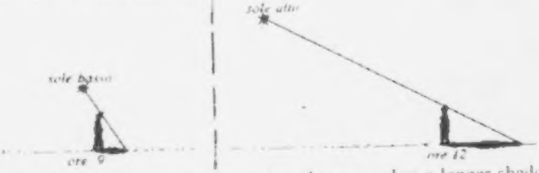
This section ended with a schematic representation of sunrays as parallels lines producing shadows of the same length wherever the nail is put.

The spatial inclination (i) of sunrays then becomes an object of study; it is unseen but repeatedly represented in the schema by plane angles seen as (iii) the intersection of directions, signalled by the usual standard sign. It is also the constant element which will explain the independence of the shadow fans from the localisation of the nail. We may note that, in spite of the analogy between graphic representations, no connection is made by students either with the situation where (iii) emerged or with the word "angle" introduced in situation c): indeed no student speaks in terms of "angle".

5 - Analysis of a short teaching sequence: maturation of "inclination"

The sequence covered 5 hours over 3 days. The initial task was as follows:

At the beginning of classwork on sunshadows, Stefano (a grade VI student) thinks that shadows are longer when the sun is higher and stronger. Other students think the contrary. In order to explain his hypothesis, Stefano produces the following drawing:



and writes: "As we can see in the drawing, the sun makes a longer shadow when it is higher, that is at noon, when it is also stronger". *We know very well that shadows are longer when the sun is lower (early in the morning and late in the afternoon). So, there is something that does not work in Stefano's reasoning. What is wrong with his reasoning, and particularly with his drawing? Try to explain yourself clearly, so that Stefano can understand.*

The aims of this task, agreed with the teachers of the class, were: to get the students to question the statement *"when the sun is higher, the shadow is shorter"*, especially as concerns the meaning of *"high"*; to develop the concept of inclination of sun rays as a tool to explain Stefano's mistake; and to question some aspects of the conventionality of the geometrical schema used by Stefano and shared by the class: particularly, the sun at the end of the segment representing the sun ray.

We may note that the expression "height of the sun" had always been used in this class without any previous questioning about its possible meanings. Some students used it with the meaning of "inclination", having in mind the apparent circular movement of the sun in the sky. Others were caught by the word and interpreted it as a "distance" (like Stefano). This ambiguity created a *complex situation*, and raised interesting problems, providing good *argumentation opportunities* which called for the (re)creation of suitable rigorous notions as well as a clarification of the crucial "variables" in the shadow schema.

In analysis of the sequence originated by the task we shall consider only the moments when the idea of "inclination" was considered and elaborated.

First individual written productions

The students recognized inclination in the shadow schema: all of them made a connection between the space situation and its plane representation.

Three students out of the 18 present that day explicitly analysed the key word "high": in order to produce arguments against Stefano's position, they referred to "inclination" as such, or compared to represented static reality (1), or to their personal conceptions about the movement of the sun (1').

Two other students used "high" as opposed to "far".

Twelve other students managed to grasp the idea of "inclination" as a crucial element in opposing Stefano's position: two expressed it only implicitly, four autonomously expressed it in explicit terms, and six did it with the mediation of the teacher.

First classroom discussion

The discussion started by focusing on the meaning of the word "high". [The concept of inclination is questioned].

A debate about the relationship between the direction of sunrays and length of shadows followed. For the first time in this discussion the word "inclination" was proposed by a student and its use was encouraged by the teacher. [This is the beginning of the work aimed at "defining" inclination (i), which goes on with differentiation of it from the height. An implicitly used entity must be focused on, named and recognised as pertinent and important for the work undertaken].

The position of sun in the shadow schema was questioned: analysing the sentence *"If I change the position of the sun, the length of the shadow changes"* that one of the students produced, some of the class discovered that the length of the shadow did not change if the sun was positioned at different points along the straight line representing the sunray. [This discovery related to a crucial property of angles - the independence of the angle from the length of the drawn sides, even if no student made an explicit connection with the "angle"].

A dynamic, global conception of the phenomenon was proposed (referring to the apparent movement of the sun in the sky). [Inclination is a variable in a dynamic situation (i')]

"Reality" came in: students recalled observations of the sun from the classroom and then one of them related "high" to the direction of the arm [probably recalling past pointing at the sun, a reference situation for (i)].

Another student claimed that *"for us, high means that the arm is high"* . Another student reacted, saying that the height of the arm is not relevant. Practically, he proposed a real-life situation to illustrate his idea - the teacher had to indicate (with his "high" arm) a far point which was lower that the point indicated by the student with his "low" arm, giving an embodied reference to inclination. [This was another important step in differentiating between "inclination" and the previous idea of "height"].

A discussion about the observed situation followed. "Direction" was explicitly associated to prolongation of the arm or drawn sunray.

Second individual written production

These students were asked to write down *"which important ideas emerged during the discussion"*. Fifteen students were present.

All the students expressed the meaning of "high": five as opposed to distance; two as inclination of the sunray (these two had already mentioned "inclination" in their first text: MI); five (three MI) as represented by the direction of the arm; four as a position in the movement of the sun (three of whom had expressed a dynamic conception in the first text). Eight students made the meaning of "high" for Stefano explicit (as "distance");

Three students characterised "direction" by prolongation of the arm (2 MI).

Twelve students referred to "inclination" in appropriate terms, associating it to the sunray, or arms indicating sun, or in opposition to the meaning of "high" for Stefano. Two implicitly worked on this concept without naming it (they both did the same in the first text). Only one student did not consider it (in his first text he had expressed this curious conception: *"the sun being high, the rays get weaker when they arrive on earth and so they make short shadows"*).

One student, Mariella, autonomously related "inclination" with "angle", although this idea did not emerge during the first discussion. One student referred to rotation of the schema (he had already taken part in the discussion along these lines).

In general, we may note that there was either a direct usage of the word "inclination" in its right meaning or a differentiation from "height" in order to overcome the previous ambiguity about these words.

Second classroom discussion

This discussion was aimed at production of a final summary text, which was to containe (according to the teacher's intentions) the meaning of "high" for Stefano, the meaning of "high" for the class, the idea that the position of the sun in the shadow schema is purely conventional and, possibly, the approach to the connection of the inclination concept with the angle concept (exploiting Mariella's idea in her second text). In the discussion, the students showed no difficulty with these goals, except for the last. Indeed relating "inclination" to "angle" was not easy: the teacher invited Mariella to present her idea about it, but for many students "angle" evoked only work on paper in the plane, and so they finally managed to "see" the connection in the shadow schema, not in space.

6. Some Conclusions

The aspects of the angle concept that we have described were of different kinds, and emerged in different ways: directly, with descriptive or interpretative aims, for static relative positions of lines (i, ii and iii), and for dynamic, consecutive positions under the effect of rotation (i' in the space and iv on the plane);

indirectely, identified by the researcher as characteristics of geometrical transformations - homothety (v) and rotation (iv').

We can identify peculiar conditions which allowed the progressive maturation of some aspects of the angle concept. We have signalled some points where the *complexity* of the situations stimulated rich *argumentative activities*. And these were crucial for the emergence of "properties", gradually preparing the ground for the angle as a geometry "object" (cf Sfard, 1997).

The studied sequence (see 5.) was not aimed at integrating the aspects of the angle concept that had worked as tools for solving problems in different situations. However, it is astonishing how often inclination (ii) and plane angles (iii) intervened in the same context, in similar situations, and how difficult they were to integrate. Referring to the analysis regarding "inclination and angle"(see 1.), we may wonder how strong the effect of embodiment is on the building of "inclination", which for so long remains an entity separated from the other aspects of the angle concept. This question raises another, more general question: if embodiment plays such a strong role in conceptualisation, what are the effects of the lack of embodiment in the traditional teaching of space geometry? From another point of view, taking into account the discussion of Euclid's definition of the "*inclination of a straight line to a plane*" (see 2.): Does embodiment constitute a negative element in the perspective of building decontextualised concepts, or rather an inevitable, and positive, phase on the road towards complex and long concept building?

The different aspects of the angle concept did not integrate, but the systematic argumentative activities made them evolve and intersect, bringing them closer each time and strengthening them. During the following activities (from April, grade IV to December, grade V) it was possible to appreciate the significant effects of these steps on the final, progressively more explicit and conscious integration of the different aspects towards the institutionalisation of the knowledge concerning angles.

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ON VERBAL ADDITION AND SUBTRACTION IN MOZAMBICAN BANTU LANGUAGES

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This report presents the results of a first interview on addition and subtraction skills of unschooled adults. The interview was carried out in ten different languages. The Mozambican languages have very regular numeration systems, based on ten. The majority of the languages also use five as an auxiliary base. The information was collected by students of the BEd program in Adult Education (Maputo) and students of the MEd program in Primary Mathematics (Beira). The student-interviewers are experienced adult educators or primary teachers and it is hoped that their present studies will contribute to the improvement of teacher education, through the richness and flexibility of oral mathematics, which is traditionally ignored in Mozambican schools.

1. INTRODUCTION

As a lecturer of Didactics of Arithmetic I dedicate quite some time to *verbal* arithmetic as distinct from *written* arithmetic. The reasons are the following:

a) Children's initial arithmetic should receive considerable attention in any course of Didactics of Elementary Mathematics. But their arithmetic is concrete, practical and verbal, just as the mathematics of unschooled adults (e.g. Carraher et al. 1988).

b) Many teachers think that mathematics always means written mathematics, neglecting or depreciating oral mathematics (Carraher et al. 1988, Draisma 1993, Gerdes 1995). But all written mathematics depends on the use of *basic facts*. These basic facts are acquired through non-written mathematics (Padberg 1986). Some teachers are not aware that, if you don't know that 15 – 7 equals 8, you cannot apply the written algorithm in order to find the 8.

c) Carraher et al. 1988 show that real life oral mathematics is in several ways superior to the predominantly written school mathematics, but its value is not recognized by the school system.

d) The present day mathematics syllabus for primary school in Mozambique recommends a considerable amount of mental arithmetic, but without explaining sufficiently that this mental arithmetic should be essentially *verbal* and not a *mental version* of the *written algorithms* (Secção de Matemática, INDE, 1989).

As a main eye-opener I have my students carry out interviews with unschooled adults on their arithmetical skills. With these interviews, I intend to achieve the following objectives:

- to confront my students with unexpected experiences;

- to collect information on verbal arithmetic done in the local Bantu languages, particularly in languages that use an auxiliary base five;
- to train my students in doing interviews with individuals on their mathematical skills and understanding.

2. THE 1991 INTERVIEWS CARRIED OUT IN BEIRA

In the end of 1991, students at the Beira Campus of the Higher Pedagogical Institute (ISP) — now *Universidade Pedagógica* (UP) — conducted a limited number of interviews with unschooled women, who were participating in a literacy program in the local Sena language. A first interview contained simple addition and subtraction problems, which according to the existing primary mathematics syllabus may be solved mentally (= orally). A second interview contained simple problems of multiplication and division.

For the students, who were all experienced primary teachers, it was a big surprise that illiterate women calculated so well in their own language, without having to write anything, being sometimes so fast that our students had difficulties in following and understanding the computations.

For me the results of the interviews were very helpful, because the women used verbal computation strategies, similar but more varied than those that are suggested in the present day syllabus for primary education. Teachers have difficulties in following the syllabus, because traditionally mathematics teaching in Mozambique focusses on written computations only (see Direcção Geral de Educação 1969, GETEA w.d., and Kilborn 1990).

With my students we analysed the particularities of verbal computation in the different Mozambican languages.

The first results were published in Draisma 1993, integrated in *A numeração in Mozambique*, a monograph on Mozambican numeration systems, organized by a group of seven ISP lecturers co-ordinated by Paulus Gerdes (Gerdes (Ed.) 1993).

The results focus on:

a) the advantages of the regular, explicit expressions used for numbers: counting after *ten* has, in all Mozambican languages, the explicit, regular form: *ten-and-one, ten-and-two, ..., two tens, two tens and one, two tens and two,* etc. There are no irregular expressions that have to be translated in terms of the known numbers, like the English words *eleven, twelve, twenty, thirty, ...*

b) particular advantages of the auxiliary base *five*, that is used in many Mozambican languages, in combination with the general decimal structure of the verbal numeration systems. These advantages are similar to those involved in the use of tiles, structured in units, fives and tens, as advocated in Japan (Hatano 1982), and similar to the advantages of the Dutch computing frame ("rekenrek"), with 20 counters, organized in two rows of ten, and in each row fives are distinguished by colour (Treffers & de Moor 1990). The similarities consist not only in the 1–5–10 structure, but also in the possibilities of interiorization of the calculation strategies.

c) verbal calculation in bases 5 and 10 may be supported by *gestures*, as the fingers of the hands present the same numerical structure, avoiding counting one by one.

d) verbal computation in a base-ten-language may make use of the advantages of the 5–10 structure, if supported by gestures, in a similar way as verbal computation in Dutch, Japanese or Korean uses a 5–10 structure, when accompanied by the computing frame (Treffers & De Moor 1990), the 1–5–10 tiles (Hatano 1982), or gesture computation (Fuson & Kwon 1992).

In Table A we present a selection of the number words used in two Mozambican languages: Tewe, a Shona variant, spoken in central Mozambique, and Changana, a Tsonga variant, spoken in parts of southern Mozambique.

Table A

n	English	Tewe (variant of Shona)	Changana (variant of Tsonga)
1	one	posi	xin'we
2	two	piri	swimbirhi
3	three	tatu	swinharhu
4	four	cina	mune
5	five	shanu	ntlhanu
6	six	tanhatu	ntlhanu ni xin'we
7	seven	nomwe	ntlhanu ni swimbirhi
8	eight	sere	ntlhanu ni swinharhu
9	nine	pfemba	ntlhanu ni mune
10	ten	gumi	khume (or chume)
11	eleven	gumi ne posi	khume ni xin'we
12	twelve	gumi ne piri	khume ni swimbirhi
13	thirteen	gumi ne tatu	khume ni swinharhu
14	fourteen	gumi ne ina	khume ni mune
15	fifteen	gumi ne shanu	khume ni ntlhanu
16	sixteen	gumi ne tanhatu	khume ni ntlhanu ni xin'we
17	seventeen	gumi ne nomwe	khume ni ntlhanu ni swimbirhi
18	eighteen	gumi ne sere	khume ni ntlhanu ni swinharhu
19	nineteen	gumi ne pfemba	khume ni ntlhanu ni mune
20	twenty	makumi mairi	makume mambirhi
21	twenty one	makumi mairi ne posi	makume mambirhi ni xin'we

28	twenty eight	makumi mairi ne tanhatu	makume mambirhi ni ntlhanu ni swinharhu
73	seventy three	makumi manomwe ne tatu	ntlhanu wa makhume ni mambhirhi ni swinharhu
89	eighty nine	makumi masere ne pfemba	ntlhanu wa makhume ni manharhu ni ntlhanu na mune
100	hundred	zana	zana

3. THE 1997 INTERVIEWS

In 1997 a group of students at the Maputo UP Campus, all linked to adult education programs, conducted the same interviews with adults in different parts of the country. Their task was to find and interview an adult with reasonable arithmetical skills, without having been to school. Until now we have received 17 interviews on addition and subtraction. The interviewed people had the following professions: peasants (8), market vendors (4), fisherman (1), tractordriver (1), housewife (1), and two non-identified professions. The interviews were held in 10 different languages. In 12 of the interviews a base 5 language was used and in the other 5 interviews a base 10 numeration system was used, including one in Portuguese.

Results — Addition and subtraction skills

All interviewed adults have more or less developed skills of verbal arithmetic: in general more developed than those of the average primary school teacher or the average UP student, who have the habit of using the written algorithms.

The interviewed adults showed a great variety of computation strategies and corresponding explanations. The adults who were interviewed rarely resorted to counting (7 out of 120 answers in addition, 19 answers out of 98 in subtraction). However, the number of cases coded as "counting" is probably too high: as the interviewers had little experience, some may not have noted the discrepancy between an immediate answer and an explanation of the answer as if it were obtained by counting.

In general the adults are sure of their computations, make very few mistakes (less than 2%), and know, when a problem exceeds their capacity.

An important aspect of the explanations given by the adults is that the explanation generally focusses on some steps of the calculation, whereas other steps remain implicit, i.e., they are not mentioned in the explanation, although they must have been executed mentally. In the presentation of results and examples of addition and subtraction, we will give examples of more abbreviated explanations (with more implicit steps) and more complete, explicit explanations.

Strategies used for addition

In Table B we present data on the different strategies that were used for a number of addition tasks. We distinguish the following strategies:

completing 10: two step addition, completing ten or a multiple of ten, like in 8 + 5 = (8 + 2) + 3 = 10 + 3 = 13

completing 5: two step addition, completing five or a multiple of five, when base five is used, as in 4 + 7 = 4 + (5+2) = (4+1) + (5+1) = 5+5+1 = 10 + 1 = 11

use 5+5 = 10: use the fact that 5+5 = 10 (or 50 + 50 = 100); e.g.: 8 + 5 = (5+3) + 5 = (5+5) + 3 = 10 + 3 (especially when base five is used, but not only)

Table B

Task: *Strategy:*	*8 + 5*	*4 + 7*	*8 + 9*	*14 + 6*	*3 + 48*	*19 + 7*	*64 + 80*	*390 + 48*
completing 10	4	8	11	11	6	15	3	6
completing 5		3		3				
use 5+5 = 10	7		1				3	
counting	2	2	2		1			
other		1			6		1	1
without explan	4	3	3	3	3	2	5	
not solved	0	0	0	0	1	0	5	10
Total	17	17	17	17	17	17	17	17

The number of cases where the interviewer got no explanation constitutes 14% of the total number of problems that were solved. These cases may correspond to the lack of experience on behalf of the interviewers, of having his interviewees explain their computations, especially, when these computations are done very quickly.

Explanation styles

Using the task 19 + 7 = ? as an example, we present two different ways of *explaining* the computation: a *more abbreviated* and a *more explicit* explanation.

Deolinda, a 54 year old peasant, explains in Changana language:

Ka *chume ni n'tlanu ni mune* ni patsa *xinwe* svi ku *machume mambirhi.*	To *ten and five-and-four* I add *one,* which makes *two tens.*
Ni tlhela ni patsa *n'tlanu ni xinwe* svi ku *machume mambirhi ni n'tlanu ni xinwe.*	Then I add again *five-and-one,* which makes *two tens and five-and-one.*

In this explanation (or is it the real computation?) the attention is fixed on the number that has to be completed (*nineteen*) and the number that is needed for that (*one,* in order to make *twenty*). Simultaneously there is a second, mental computation going on: the *one* that is needed, is taken from the *five-and-two (seven),* remaining *five-and-one,* to be added later to the *twenty.*

Beatriz, market vendor, 60 years, explains more completely, in Gitonga language:

Nhididussa *muéyó* hava nha *libandre na dzimbili,* nhipata hava nha *likumi na libandre na dzina,* para gukala *magumavili.*	I took away *one* from the *five-and-two,* added it to the *ten and five-and-four,* in order to make *two tens.*
Depois, nhipata *magumavili* ni *libhandre na muéyó,* dzingatsala hava nha *libhandre na dzimbili.*	After that I added *two tens* with *five-and-one* that had remained from the *five-and-two.*
Gukala *magumavili na libhandre na muéyó.*	That makes *two tens and five-and-one.*

In this explanation, the calculation starts with the origin of the *one*, that is needed to make *twenty* from *ten-and-five-and-four.* In order to take *one* from *five-and-two,* you must first have heard, that *one* is needed. In this case, the explanation starts with a secondary calculation (7 – 1 = 6), then going back to the main calculation (19 + 1 = 20), that had already been done mentally, before doing 7 – 1 = 6.

Strategies used for subtraction

In Table C we present data on the different strategies that were used for a number of subtraction tasks. We distinguish the following main strategies:

one step subtraction: subtraction, from the position where that is possible, like 14 – 5 = (10 – 5) + 4 = 5 + 4 = 9

two step subtraction: subtraction, like 14 – 5 = (14 – 4) – 1 = 10 – 1 = 9

use 5+5 = 10: use the fact that 5+5 = 10 (or 50 + 50 = 100)

Table C

Task: *Strategy:*	*11 – 4*	*14 – 5*	*12 – 7*	*16 – 9*	*62 – 5*	*31 – 28*	*910 – 79*
one step subtraction	4	6	4	7	7	2	1
two step subtraction	7	6	5	5	4	1	
use 5+5 = 10			1				
counting	5	4	5	3		2	
other					2	7	2
without explanation	1	1	2	1	1	2	
not solved	0	0	0	1	3	3	14
Total	17	17	17	17	17	17	17

Explanation styles

Using the task 62 – 5 = ? as an example, we present two different ways of explaining the computation: a more abbreviated and a more explicit explanation.

Helena, a 43 year old housewife and market vendor explains in Changana language.

after having given the answer rapidly:

Nisuse *ntlhanu* ka *ntlhanu wa makume ni lin'we,*	I took away *five* from the *five tens and one* (ten),
se niteka le ta *timbirhi,* nipatsa ka *ntlhanu.*	I took those *two,* added to *five,*
se sviku *ntlhanu wa makume ni tlhanu ni timbirhi.*	that makes *five tens* and *five-and-two.*

Note that in Changana the expression for 60 is *ntlhanu wa makume ni lin'we,* which means "five tens and *one*". You can hear that the *one* is a *ten,* because of the prefix li– used in *lin'we.* The numeral for 51 is slightly different: *ntlhanu wa makume ni xin'we,* meaning also "five tens and one"; in this case people understand that the one refers to one thing (*xilo xin'we*). However, in order to avoid ambiguity, you could say explicitly *ntlhanu wa makume ni chume lin'we,* that is "five tens and one ten".

Faina, a 65 years old peasant, explains more explicitly in Sena language:

Pinthu *makumatanthatu* bulusa *nkhumi ibodzi,* anasala *makumaxanu.*	*Six tens* of things, take away *one ten,* remain *five tens.*
Pa *nkhumi ibodzi* bulusa *pixanu* pinasalambo *pixanu.*	From *one ten* take away *five,* remain *five.*
Nda kuata pinthu *piwiri* thimizira pa *pixanu* pinankhala *pinomwe.*	I take *two* things, add to *five* and obtain *seven.*
Makumi maxanu na pinomwe nde pinthu *makumi maxanu na pinomwe.*	*Five tens and seven* is *five tens and seven* things.

Note that Faina's last step is no step at all: she just repeats the three parts of the numeral *five tens and seven* : a typical example how the regular verbal numerals simplify the computation.

Faina uses sometimes the word *pinthu,* which means *things,* possibly because it was used by the interviewer, when posing the problem. On the other hand, Faina uses the numerals always with the prefix *pi–,* what is probably an indication that she has the word *pinthu* in mind, although it may have also a kind of neutral, abstract meaning, representing a way of calculating without thinking about specific objects.

Some interviews support the idea of Reed & Lave 1981, as quoted by Carraher et. al. 1988, that the verbal calculations are done as manipulation with quantities, as the numerals for 1, 2, ..., 9 function as adjectives, that require a prefix that varies with the objects they refer to. However, in general people use the prefix that corresponds to the word for "thing", which turns the verbal calculation abstract, like manipulation of *verbal symbols.* The difference with abstract written computation is that in verbal computation the tens, hundreds, thousands, etc. are explicitly mentioned, whereas in the written standard algorithms only basic facts are used.

4. PERSPECTIVES

Presently, several MEd (Primary Mathematics) students at UP Beira are writing their final thesis on research done in verbal mathematics in different Mozambican languages, with children or unschooled adults.

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TEACHERS' BELIEFS AND THE 'PROBLEM' OF THE SOCIAL

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Research on teachers' beliefs has tended to view these as stable systems which operate consistently across contexts. This paper describes a longitudinal study which tracked a cohort of preservice mathematics teachers through their mathematics method course and into their first year of teaching. It takes a single case in order to show the variation of commitment across context and suggests a way of describing teachers' beliefs so as to recognise the interrelationship of subjectivity and social context.

In 1992 Paul Ernest wrote a paper entitled "Constructivism and the Problem of the Social" in which he pointed to the limitations of constructivism (in particular radical constructivism) in explaining the transmission and acquisition of mathematical practices in social context. Children may well construct their own knowledge, but not in circumstances of their choosing. They may not have the same access to the pedagogic texts which are on offer, as Cazden and Mehan (1989) for example point out. The organisation of classrooms, the flow of questioning and discussion and the formation of groups can distribute very different kinds of pedagogic knowledge to different categories of children.

My interest in this paper is not in constructivism but rather with the 'problem' of the social in relation to another predominant focus in the field of mathematics education, that of teachers' beliefs, and the relationship between teachers' beliefs and classroom practice. For both preservice and more experienced teachers, the argument flows, teachers' beliefs either do, or should, regulate the way in which they teach (for a review, see for example Thompson, 1992). This interest continues to generate a significant body of research and in my paper I want to consider the methodological implications of some of this and in particular its failure to theorise the social. My paper is interrogative rather than programmatic, and the study I draw from was designed for an interest other than teachers' beliefs, but it does, I think, raise interesting questions which that research perhaps needs to address.

Underpinning most of the research on teachers' beliefs is the assumption that beliefs are individually constructed and owned, stable systems that organise action across a range of contexts. These stable systems or frameworks are organised by a single, stable rationality which is assumed to operate consistently across contexts and which regulates action. In other words, it is assumed that there is, or at least should be, a correspondence between cognition and action such that cognition organises action. It follows therefore that if teacher X holds to a quasi-empiricist view of mathematics, this should be reflected in the way in which he/she organises her classroom, arranges mathematical tasks and so on, and also in the way he/she talks about teaching either in an interview or through a questionnaire.

This rationalist account of the individual, operating selectively but nevertheless consistently on the world, reproduces what Henriques et al (1984) term the individual/social dualism, that which holds society and the individual as 'two entities [..] necessarily thought of as antithetical, as exclusive (though interacting), as separable and even as pulling in opposite directions' (pg.14/15). This dualism rests on an untheorised 'core, presocial individual which remains intact' as it moves across contexts (pg. 21) 'who is the actor who takes on and performs the roles. [..] The external has been welded on to the individual but is still peripheral, able to be taken on and cast off' (pg. 23).

This approach has important implications for research. Insofar as it foregrounds continuity of discourse and practice, research effort must inevitably be driven towards seeking out this continuity. In relation to preservice teacher education and beginning teaching, for example, which is my own area of interest, studies tend to focus on whether, and the extent to which, student teachers change their beliefs and attitudes over the course of their study, whether these new frameworks then 'transfer' consistently into student teaching practice, and again whether continuity is further detected as students become beginning teachers. Instead of continuity, however, the studies with few exceptions find disjuncture. Students' or teachers' classroom practice is found to be largely inconsistent with their espoused beliefs about teaching and the effects of teacher preparation seem to be 'washed out' (Zeichner & Tabachnik, 1981) by beginning teachers both in words and in deeds. (See for example Tabachnik & Zeichner (1986) Borko et al (1992),Eisenhart et al (1993))

Little attention is given to the status of the texts which form the basis of data analysis in these studies. Research on teachers' beliefs uses interviews, classroom observation and questionnaires, in different combinations and with different emphases. In most cases, these are regarded as transparent and seamless texts, as windows to that unified rationality which is assumed to be operating across space and time. Rarely is self-reporting interrogated, or the strategies interviewees use when positioned within these contexts. Variations in account by respondents at different points in time, or in different locations, are foregrounded only insofar as these constitute a "problem", as a deviation from the normality of the consistent narrative.

This discontinuity is not explained by re-examining the founding assumptions about subjectivity nor the interrelationship between subjectivity and context. Rather, the students or teachers themselves are held to account (their beliefs did not really change during the course of study after all or if they did, were faulty) or the programme of study is considered deficient (it failed to effect the desired change in beliefs). Very often the teaching context is evoked to explain disjuncture; the lack of support from other teachers, the pressure of time in covering the syllabus, the social organisation of schooling, and so forth. As Scheffler comments:

> with independent knowledge of the social context, we may judge belief as revealed in word and deed. Where the latter two diverge, we may need to decide whether to postulate weakness of will, or irrationality, or deviant purpose, or ignorance, or bizarre belief, or insincerity; and the choice may often be difficult (Scheffler, cited in Thompson, 1992, pg.134)

I want to suggest that this 'divergence' can be read differently. I argue in this paper that human subjects are inserted into different social activities and that what they "believe" is contingent and not necessarily stable across these. While human subjects do acquire repertoires of knowledge and skills and more or less loosely structured webs of predispositions or commitments to act, which one might call "beliefs", these are foregrounded and backgrounded according to the context in which subjects operate. Arguing that beliefs are stable across contexts leads us into many of the same difficulties we have encountered in researching the relationship between mathematics in schools and everyday life. Both Lave (1988) and Dowling (1997) have argued that this relationship is characterised by disjuncture rather then continuity, and develop their descriptions through an elaboration of the notions of site, context, and most particularly in the case of Dowling, subjectivity. Unfortunately there is no space here to develop these ideas but I hope to illustrate some of these issues by drawing on data from a study concerned with preservice mathematics education and beginning teaching.

The study

The data which I refer to in this paper was drawn from a two year longitudinal study which tracked 23 preservice mathematics teachers-to-be through their university mathematics method course, and seven of this group as beginning teachers into schools. Data gathering in the first year incorporated observation notes of sessions, reflective journals, teaching practice journals, curriculum projects, tests and exams, and interviews with students at the end of their course. In the second year of the study, I visited each teacher four times, combining interviews with videoed lessons. My focus is the recontextualising of pedagogic practice; what students acquire on their method course, and what they recruit into their classrooms as beginning teachers.

The analytic framework I use is drawn from Dowling's social activity theory. Following this, I describe mathematics education as a social activity, realised in two subactivities, mathematics teacher education and mathematics classroom teaching. As a social activity, mathematics education is interested in the transmission and acquisition of mathematical practices, and both produces and reproduces itself through texts and subjectivities, in specific contexts. Both mathematics teacher education and mathematics teaching are specialised in turn by specific discourses; constructivism, fundamental pedagogics and so forth, distributed to particular social positions; teacher educator, student, teacher and pupil. The texts I shall examine here (re)produce both mathematics teacher education and classroom teaching; the first through the practices of an acquirer, the second through the practices of a classroom teacher. I want to focus on four texts produced by a student teacher, Mary; an extract from her reflective journal, an extract from her teaching practice journal and an interview she had with me at the end of her preservice year, and finally an extract from an interview with me as a beginning teacher.

Methodological issues

The texts I shall refer to involve accounts of self and settings, of classrooms, staffrooms, lecture halls, and moments of private reflection. What is the status of these accounts? Clearly they do not straightforwardly provide windows on different worlds, be they inner or outer ones. Rather, they represent these worlds, not necessarily consistently, through selective description and redescription.

I argue that in each case the produced text is evoked within a particular context, by a specific invitation to speak. In this sense the contexts are productive. At the same time they are constraining insofar as each context, with its audience, both canalises and silences expression. Each context is an invitation to subjects to position themselves in relation to each other, and recruit or recontextualise linguistic and somatic resources in order to achieve this. The evoking context in each case foregrounds and backgrounds subjectivities, repertoires and positions and in this way motivates the selective recruitment of resources. In other words, each context calls forth or interpellates certain subjectivities and backgrounds others, and these subjectivities recruit resources in their elaboration. Potential resources for recruitment are learning theories, utterances of lecturers, teaching colleagues and learners, ensembles of voices and associated practices in various sites, mathematics and so on.

The reflective journal

Students were required to write a reflective journal as part of the requirements for their mathematics method course. An extract from Mary's first entry is as follows:

> The first session left me excited, stimulated, enthusiastic and eager to do a lot of thinking and exploring so as to be maximally effective and creative. I'd like to be holistic in my teaching

> approach to my subject, using all that I am - social, emotional and intellectual - to facillitate (sic) a total learning experience for my pupils.

Mary here locates herself as a student and teacher-to-be in relation to her audience, the teacher educators with whom she affiliates. She achieves this affiliation in the following ways: by invoking an inner voice, one immediately made public through introspection, one engaged in a project of self transformation which elsewhere in the journal she asserts as the key to successful teaching practice. In articulating this she is privileging the reflective practice emphasised on the mathematics method course. In other journal entries she affiliates by partitioning teaching practice in the way this is done on the course, contrasting 'teaching with an exam orientation' and its attendant emphasis on achievement and the learning algorithms, with the provision of a 'creative, holistic experience' which privileges understanding, pupil awareness and discovery of concepts. She emphasises the importance of demonstrating the usefulness and logic of mathematics to pupils instead of rule-based presentations, of 'exploration' rather than 'spoonfeeding'; of guesses and haphazard explanations which are recognised and rewarded rather than 'only one answer which the teacher has', of making use of pupil's own intuitive methods against the safety of the textbook, the syllabus and tried-and-tested methods. She celebrates the development of intuition, investigating, exploring, questioning, making observations and recording discoveries, approximating those engaged with by "real mathematicians", over the traditional, algorithmic methods she was exposed to as a pupil. In all these ways she espouses a commitment to the precepts valued on the course and she does so to achieve an affiliation with her teachers, the teacher educators.

On the basis of her journal, one could conclude that Mary's "beliefs" were very close to those espoused by the teacher educators on her method course.

The teaching practice journal

Towards the end of her second teaching practice, which takes place in the second of the two-semester course, Mary wrote in her journal as follows:

> I learnt several helpful things from the subsequent discussion with Mrs B [the supervising teacher] and V [a fellow student]:
> - my explanations are clear, precise and show an understanding of where the pupils are at and how they will best understand
> - I demonstrate an awareness of pupil misconceptions and plan for it in my lessons, evidence of a good teacher
> - I need to be 'harder and meaner' as far as discipline is concerned
> - my pupil interaction, individual explanations and rapport are very good.

She goes on to say that she in future she must develop order and system in her teaching in the form of systematic and orderly boardwork and regular and thorough marking of homework. She needs to emphasise to pupils that work must be meticulously set out and very systematic, 'exactly as the memorandum would require', assigning marks for various steps, neatness and the final answer. She discusses various penalties that might be imposed for the non-production of homework.

Here, Mary's position would appear to be at odds with the commitments of her reflective journal. In her teaching practice journal, she addresses a different audience, her teaching practice supervisor (either a university academic or someone appointed by the university to assist with teaching practice supervision but only co-incidentally a specialist in the students' own subject area), and here she recruits those resources which will establish her competence; a careful exponent of mathematics, monitor of pupil's efforts and a relatively strict regulator of classroom behaviour.

The final interview as a student

In the interview with Mary towards the end of the HDE year she adopts a stance primarily with respect to the researcher and not directly in relation to the lecturers. She does not affiliate with the lecturers, but objectifies them and acts selectively on their pedagogic message.

> I think that's definitely one thing that has come out this year, anti the teacher feed, you know, everything into them and very much pupils participate, do the work, find out things for themselves, a lot of pupil interaction and pupil discovery and being open-minded, thinking about your subject and not taking the syllabus for granted, making it relevant to, like in maths to what the children need, where it links with the outside world and that kind of thing, [....] is it relevant, can I teach it in a relevant way, not in an isolated, unconnected kind of way

Having said this, she also comments that 'a lot of things I think I have accepted I'm not sure that I'll actually implement [...] I found [on teaching practice - PE] there was quite a discrepancy between what I was writing in the journal and how I was teaching in some ways.' Mary recruits the practices of teachers she encountered on teaching practice to raise questions about the applicability of what she has learned on the mathematics method course. She wants to be innovative but as one of the teachers on her second teaching practice pointed out, 'it is the old methods that work, the old methods get the results.'

She thus affiliates with the lecturers insofar as they provide useful orientations to, and resources for, teaching, but she affiliates also with classroom teachers. She indicates that there are aspects of the way she was taught as a pupil that she regards as important; orderliness and structure, meticulous boardwork, setting and checking of homework. She acknowledges that there may well be a contradiction here between what she holds to be important for mathematics learning but she thinks that this structure is important in mathematics teaching and learning.

Talking as a beginning teacher

In an interview in the first semester, Mary spoke in the following way about classroom teaching:

> that's in a way I think my frustration a little bit because I'm doing the same thing every lesson with every class .. do an example on the board, let them do some work, you know, give them another example and let them do some more work, mark some homework, do an example, do some work, and that's how it is every time

Discussions about classroom teaching were imbued with concerns about discipline and control, being able to get through the syllabus at an adequate pace and explain mathematical content carefully and clearly. The only aspects of the method course which Mary indicated that she used were a small number of discrete activities which she had engaged in on the method course. Apart from this, she has not recruited anything else.

Conclusion

On the face of it, these extracts suggest that Mary's 'beliefs' about teaching have changed in the course of writing her reflective journal, her teaching practice journal, speaking to me in her interview while a student, and then again with me as a beginning teacher. We might conclude from this, following research in the past, that Mary's beliefs did not really change and that her writing in the reflective journal was simply a veneer, or a form of 'strategic compliance' (Lacey, 1977). Or we might conclude that Mary authentically 'believed' in the views expressed in her journal, but that these changed when she moved into teaching. The problem is, on what basis do we privilege one reading over another? To question her again and again in search of an unvaried "authentic" account would be to engage in infinite regress.

It is perhaps more productive to view Mary as a human subject inserted into a range of different contexts, each of which defines competence differently. In other words, each is characterised by different evaluative conditions. Within each she recruits what is appropriate in order to establish herself as competent. She uses the reflective journal to restate the pedagogic message privileged on the method course and demonstrate her commitment to self transformation and good pedagogic practice as defined by the lecturers on the course. In this she locates herself as a successful acquirer. She uses the teaching practice journal to establish competence in another

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environment, that of the school classroom with respect to pupils and other teachers, being able to exercise control and explain carefully. Here she is positioned as a competent transmitter, as defined in the school environment in which she is placed. There is a resonance between her interview with me as a teacher and her teaching practice journal writing as a student teacher, achieved in that both address the same site of practice and the same social relationships. Disjuncture appears between these inscriptions and those of the reflective journal, as the latter is produced in a different site ordered by different social relations.

By foregrounding the differences in social context it is possible to describe Mary's multiple positioning in positive rather than negative terms, and to construct a rationality for her actions and utterances which is both multiple and contingent.[1]

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[1] It becomes possible, also, to consider the intersection of social structuring on the basis of ethnicity and gender with that of context, and describe how the male, and black students, of my sample position themselves differently to Mary as acquirers, and again as teachers. We are then able to prise open not only the concept of 'belief', but that of 'the teacher' as well.

FROM NUMBER PATTERNS TO ALGEBRA:
A COGNITIVE RELECTION ON A CAPE FLATS EXPERIENCE

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Abstract

This paper explores some of the cognitive difficulties with the generalized number pattern approach to algebra experienced by junior secondary students at a typical Cape Flats school. Three patterning activities were examined: a worksheet; a physical match stick pattern; and a functional table. Each activity engaged the students in: the "seeing" of the pattern; expressing the pattern in words; expressing the pattern a concise (symbolic) form; and solving problems based on the pattern. A number of cognitive difficulties were uncovered, e.g., a fixation with recursive rules; a tendency to over-generalize; a piece-meal perception of the pattern; inadequate linguistic facilities, etc. The results imply that these cognitive obstacles have to be overcome before the typical junior secondary student on the Cape Flats will be able to make the jump from number patterns to elementary algebra.

Background

The generalized number pattern approach to algebra has become a prominent part of the curriculum reform debate in South Africa (e.g., AMESA Western Cape Region Inservice Curriculum Material for Algebra, 1995; Draft Syllabus of Western Cape Education Department, 1996). While the existing literature deals extensively with the mathematics beyond number patterns (e.g., Abbot,1992; Andrew, 1992; Andrews, 1990; Pagni, 1992; Richardson, 1984), the factors that might influence the overall viability of the generalized number pattern approach in the classroom, e.g., cognitive demands, feature only in a few papers (e.g., Booth, 1986; 1988; MacGregor and Stacy, 1993; Orton and Orton, 1994; 1996). The viability of the approach seems to be a tacit assumption that runs throughout the literature. Booth (1989) and Pegg and Redden's (1990) discussions of guiding questions that are to be asked during the approach could even be interpreted as guidelines for a possible instructional methodology. This paper reports on a small scale research project aimed to test the viability of the approach with junior secondary students at a school on the Cape Flats.

Research Methodology

Grade 8 and 9 students were used in the research as they constitute the Junior Secondary Phase. Based on the guiding questions suggested by Booth (1989) and Pegg and Redden (1990), number patterns were used as stimuli to engage pupils in the following activities:

1. Experiencing ("seeing") number patterns.
2. Expressing the rules which govern the particular number pattern in full sentences.

3. Rewriting the rule(s) which govern the number pattern in an abbreviated form.
4. Using the pattern to solve the problems more efficiently.

Only students who volunteered to be interviewed were used for the research. The interviews were conducted during the activities and were audio taped with the students' consent.

The research involved three patterning activities:

Activity One: The Worksheet

The first activity was aimed at providing an overview of how the students would react to a number pattern when presented in the form of a worksheet. The drawings of the first three match stick figures in the worksheet are shown in Figure 1.

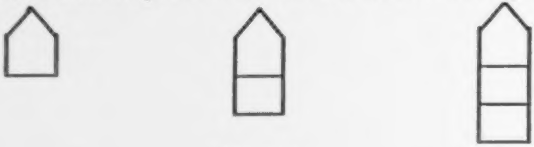

Figure 1

The worksheet was given as a take-home exercise which the students had to fill in and return at a pre-set date. Three volunteers were interviewed afterwards to clarify some of their responses to the questions, providing useful insights into their thinking and revealing some cognitive difficulties.

Activity Two: The Match Stick Pattern

The second activity dealt with a deeper study of the students' cognitive processing when confronted with a physical match stick pattern of which the interviewer physically built the first three figures (Figure 2) while the students watched.

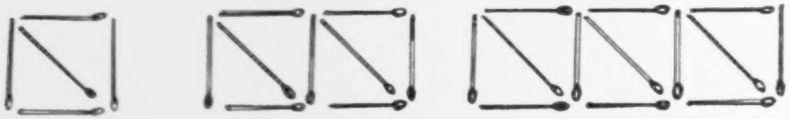

Figure 2

The students were then invited to continue the pattern and were interviewed during the activity in an attempt to explore the cognitive processing beyond their responses to the set of guiding questions as suggested by Booth (1989) and Pegg and Redden (1990).

Activity Three: The Functional Table

The third activity dealt with a number pattern presented in the form of a function table (Table 1).

Top Row	1	2	3	4	5	6	...
Bottom Row	2	5	8	11	...	...	...

Table 1: *Function Table for Activity Three*

Students were told that the table works like a computer that uses a "secret rule" to convert the numbers in the "top row" into numbers in the "bottom row". Reference to "x" and "y" to indicate independent and dependent variable respectively was deliberately avoided in order to allow for the creative invention of their own representation system rather than imposing pre-set symbols (Reynolds and Wheatley, 1994) on them. Once more the students were interviewed during the activity.

Results and Discussion

Activity One: The Worksheet

Table 2 provides a summary of the grade 8 students' responses to the questions in the worksheet. The cognitive requirements of the individual questions and processes beyond some of the responses will be discussed.

	Question 1	Question 2(a)	Question 2(b)	Question 3	Question 4
Correct	97,9 %	66,7%	33.3%	20,8%	52,1%
Wrong	2.1%	31,2%	64,6%	72,9%	43,7%
No answer	0%	2,1%	2,1%	6,3%	4,2%

Table 2: Grade 8 students' performance in the worksheet.

The questions in the worksheet read as follows:

Question 1: *Fill in the following table (Table 3) for the first six figures in the sequence.*

Position of figure in sequence	1	2	3	4	5	6
Number of matches	5	8	11	...	...	...

Table 3: Number of match sticks in each figure in the worksheet pattern

The question posed little challenge other than the spotting of a recursive rule (add three matches every time). Most students were able to do this successfully.

Question 2(a): *Find a way to work out how many matches you need to make up the 17th figure in the sequence?*

Question 2(b): *How many matches do you need to make up the 57th figure in the sequence?*

Both Questions 2(a) and 2(b) can be solved by adding on three matches until the 17th and 57th figure respectively are made up. The decline in the percentage of correct answers shows that the process becomes increasingly exhaustive the further down the figure is situated.

Question 3: *If you have not done so yet, work out a general formula that you can use to determine the number of matches in any one of the figures in the sequence. Apply this formula to the n-th figure.* Success in this question requires of the solver to reason beyond a recursive rule in order to make a generalization and then to translate it into a symbolic representation. At least two mental frames are required: one to perceive a generality in the pattern, linking the number of matches in any one figure to its position in the row; and another frame for the translation of this generality into an appropriate algebraic model, using some form of symbolic notation. If the solver does not posses such frames at this stage, this activity should create the need to develop them. The question elicited four types of responses:

(1) *The arithmetic operation, e.g. 26 + 3 = 29.* Many of the responses included this particular example. The interviews revealed that the *n-th* figure referred to in the question was read as a first letter abbreviation for the *ninth* figure, which links this type of response to the initial letter response. Hence 26 which is the number of matches in the 8th figure plus 3 to get the number of matches in the 9th figure. None of the pupils explored the other possibilities, e.g., 19th, 90th, 99th, etc.

(2) *The "add three every time" response written out in full.* Crude as it may be, it gives an honest account of the recursive rule that many have used.

(3) *The symbolic formula, e.g.,* $n \times 3 + 2$ *and* $n + 3$. The response $n \times 3 + 2$ was supposedly an indication of the successful attainment of a connection between the variable n and the position of a figure in the row, which is a prerequisite for the discovery of the formula. The response $n + 3$ shows that the respondent interprets n as the number of matches in any given figure rather than as its position in the sequence; hence the argument that $n + 3$ must be the number of matches in the consecutive figure. This is the same as the recursive formula $f(n+1) = f(n) + 3$.

(4) *The initial letter response where n is interpreted as an initial letter abbreviation for any word that starts with n, e.g., "nine" or "north".* This response is probably the result of the premature imposition of a conventional symbol (*n*) for the variable via the question, before the need for it has arisen spontaneously. On the other hand, it could also be the result of prior exposure to examples in arithmetic where initial letter abbreviations are indeed desirable, e.g., *l* for liters; *s* for seconds; *m* for meters; etc.

Question 4: *In what position in the sequence would you find the figure made up of exactly 98 matches?*

In this question there was a significant increase in the number of correct answers. This revealed that when the recursive rule is applied, it is easier to find the position of the figure made up of 98 matches (the 32nd figure) than it is to find the number of matches in the 57th figure. Almost all of the responses to Question 4 were single number answers with no indication as to how they were worked out.

A fixation with a recurrent rule turned out to be one of the major cognitive difficulties preventing successful pattern perception and rule generation. It turned out to be a cognitive obstacle for a number of students in all three phases of the research. For example, Tracy, 13 years old, explained:

4) T: Just add three every time.
5) I: How did you know to do that?

6) T: Five... three... You must add three over there to get eight... and eight plus three to get eleven... eleven plus three to get fourteen... fourteen plus three to get seventeen.
7) I: Right... Uhm... When you did that, did you look at the pictures or did you look at the table?
8) T: At the table.
9) I: So you never had a look at the pictures actually?
10) T: No.

Activity Two: The Match Stick Pattern

This activity revealed a number of cognitive difficulties that were picked up in the interviews. These include:

(1) *A fixation with a recursive rule that prohibits the generation of a functional relationship between the number of matches in a figure and its position in the row.* Students consistently failed link the number of matches in a figure to its position in the sequence rather than to its predecessor. The situation was probably aggravated by having them physically build the match stick figures. In doing so, it soon becomes clear that the next figure in the row can be constructed by adding four matches onto the previous one in a particular configuration. Although this is important to know, pupils al too soon became trapped in this discovery and failed to make the second and more important discovery -- the functional relationship between the figure and its sequential position that is to become the general rule.

(2) *Over-generalization from the known to the unknown.* This is based on the assumption of a direct proportionality between the number of matches in a figure and its position in the sequence. For example, if the 10th figure consists out of 41 matches, then the 80th figure would consist out of 41 x 8 matches.

(3) *Using repeated addition as an alternative for multiplication.* This complicates the formulation of a generalized rule that links the number of matches in a figure to its position in the sequence and totally eradicates any hint of a correct symbolic version of such a rule.

(4) *Impulsive multiplication or division of the first two discernible numerals.* This strategy probably stems from: (i) earlier learned frames in arithmetic where a good strategy usually amounts to the careful selection of one of the four basic arithmetic operations and its application to two given numbers; and/or (ii) a recognition that the pattern is expanded by successive additions of four matches at a time – neglecting the fact that the first "block" in the pattern consists out of five matches.

(5) *Problems with the expression of a generalized mathematical rule in a natural language.* To compensate, the application of the rule is demonstrated by using specific examples (often only one) and hence loosing the generality. For example Ester, 15 years old, who explained:

58) I: Okay... Can you write down a general rule?

59) E: A general rule?

60) I: Yes... Write down the method so that you can use it for any pattern.

She then wrote: "If I want to get up to 27 patter (*sic*) I say 5 match stick (*sic*) plus four plus anather (*sic*) four up antill (*sic*) I get to 27 anthen (*sic*) I get my answer". Note how she too used repeated addition in preference to multiplication. One would expect Ester to have come up with a general rule description like: Take five and keep on adding fours to it until you have added one less fours than the number of the 'pattern'. Yet she chose to explain her rule by using the 27th figure as an example.

Activity Three: The Function Table

This final activity revealed a number of cognitive difficulties in addition to those already discussed. These included the following:

(1) *A piecemeal perception of the pattern leading to inadequate generalizations or generalizations unfit for translation into the algebraic code.* Some students failed to take the whole pattern into account, choosing rather to focus on a small part of it only.

(2) *Failure to spontaneously check the validity of their assumed rules.* This lack of metacognitive awareness was prevalent amongst all of the interviewees. They simply made certain assumptions about the pattern without stopping to check its validity. This led to, for example, over-generalization going undetected.

(3) *Immature ways of expressing arithmetic operations.* For example, Claude, 14 years old, who also noticed a recurrence in the "bottom row". However, instead of seeing it as the recurrent addition of three every time, he sees it as "two numbers gone in between". This is how he explains his perception of the pattern:

5) C: With the 'bottom row'...uhm... there's two numbers gone like in between the... the numbers...

6) I: Explain that again.

7) C: The 'bottom row' is like... is two numbers... uhm... two numbers gone in between the numbers here like... 2, 5, 8, 11... between 2 and 5 there must be 3 and 4...

....

14) I: Can you explain how you got to 14 and 17?

15) C: Here in between 2 and 5, 3 and 4 is gone sir... like here between 5 and 8, 6 and 7 is gone... and the same here by 8 and 11 sir... so it will be the same here by this two... 14 and 17.

Claude's rule is the same as adding on three every time, provided that one works with consecutive natural numbers, but the problem with such a crude description is that it would be very difficult to translate directly into algebraic form. One would have to convert it into proper arithmetic form first, and then perhaps into algebraic form.

(4) *The cognitive need for a more succinct symbolic representational system has not yet matured.* None of the interviewees spontaneously attempted to use some form of symbolic representation for the variable. Instead all chose to explain their methods by means of numerical examples, in some cases more than one example to show how the method can adapted to suit different numbers in the pattern. Symbolic representations were only used when specifically asked for during the interview, and then in a clumsy manner, not doing justice to their verbally stated general rule.

No significant differences between grade 8 and grade 9 students' responses were detected. They all struggled with the same cognitive difficulties.

The cognitive difficulties with the number pattern approach that were highlighted by this research are not necessarily linked to problems with algebra as such, but rather to pattern perception and representation. If the number pattern approach is to succeed at all, priority will have to be given to the addressing of these cognitive difficulties.

Conclusions

The cognitive difficulties that Cape Flats students experience with the generalization of number patterns are by no means unique to this area. Researchers from as far south as Australia (MacGregor and Stacy, 1993; 1994) and as far north as England (Orton and Orton, 1994; 1996) have reported similar findings. What this research points out, however, is the need for some form of cognitive intervention to equip the Cape Flats junior secondary student with the necessary skills to transcend the cognitive difficulties that were pointed out. An adaptation of the existing methodological framework of the generalized number pattern approach to include opportunities for the much needed cognitive intervention is not only necessary, but also seems more feasible than a separately taught intervention programme.

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AFFECTIVE DIMENSIONS AND TERTIARY MATHEMATICS STUDENTS[1]

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Abstact

A large sample of Australian tertiary mathematics students enrolled in the first year of their undergraduate courses was surveyed. Background information was gathered, as were the students' attitudes and beliefs about factors associated with explanations for gender differences in mathematics learning outcomes. We report findings from the analyses of the various affective measures by gender, socio-economic status, language background and age group of students. Several significant differences were found for each grouping variable, particularly gender and age. Analyses of two-way interactions revealed that language background was a critical intervening variable. The results suggest that students' attitudes and beliefs are influenced by a complex of interacting social factors.

Introduction

For some time it has been recognised that participation rates in mathematical studies at all levels are not equitably distributed among groups within populations. While the stereotyped image of mathematics as the domain of white (Anglo-Saxon), middle-class males has been challenged, data reveal that disadvantaged groups tend to be under-represented. Almost without exception females are found less likely than males to study the most demanding mathematics courses offered and to persist with mathematics to the highest degree levels (Leder, Forgasz & Solar, 1996). The identification of other disadvantaged groups would appear to be context dependent and may be based on racial, religious or socio-economic differences, for example.

Using a variety of methods and theoretical perspectives, previous research exploring gender differences in mathematics learning outcomes has identified a complex of contributing variables: cognitive, affective, and environmental or contextual (e.g., Burton, 1990; Fennema & Leder, 1994; Leder, Forgasz & Solar, 1996). The recently published *Third International Mathematics and Science Study* [TIMSS] data showed that Australia was one of only a few countries for which there were no significant gender differences in the mathematics achievement of the population of 13-year olds tested (Lokan, Ford & Greenwood,1996). Analyses of the achievements of grade 12 mathematics students in the state of Victoria have revealed that performance differences relate to the form of the assessment; males tended to outperform females on timed examinations while females' performance levels were higher on more open-ended classroom (and take-home) tasks (Leder, 1994). Kimball (1989) similarly reported that the direction of gender differences in mathematics achievement is related to assessment type. Australia-wide, males' participation rates in the most demanding

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mathematics options at the grade 12 level are more than twice that of females' (Leder & Forgasz, 1992). Hence, participation rates are an important focus of research endeavour in Australia. Much Australian work has centred on school-aged children, with factors influencing participation rates in mathematics and learning outcomes at the tertiary level attracting little attention.

Higher participation rates in school mathematics in Australia have been found for students from higher socio-economic backgrounds [SES] (Ainley, Robinson, Harvey-Beavis, Elsworth & Fleming, 1994; Teese, Davies, Charleton & Polesel, 1996). Lamb (1997) reported that gender differences in participation rates were associated with the interaction of positive attitudes and beliefs about mathematics and SES. At the tertiary level, Forgasz (1996) found that the attitudes and motivations of mature-age[2] mathematics students were more functional (that is, more likely to lead to success and future participation) than those of their school-leaver counterparts and argued that more mature-age students should be accepted into tertiary mathematics courses. Perceived levels of discrimination by gender and ethnicity, Forgasz (in press) contended, had the potential to impact negatively on the decisions of some students to persist with tertiary mathematics study.

In this paper, we examine differences in the attitudes and beliefs of tertiary mathematics students by gender, SES, language background, and age. We also explore for interaction effects that might add to our understanding of gender differences in mathematics participation rates. Of special interest also were differences between school leavers and mature-age students.

The sample, instrument and methods

A survey questionnaire was administered to mathematics students enrolled in their first year of undergraduate study at five Australian tertiary institutions (see Table 1). The institutions were carefully selected to reflect the diversity within the Australian higher education sector; established and newer universities were represented as were those offering traditional academic courses and more vocationally oriented programs.

A survey questionnaire which aimed to identify critical variables that might influence students' decisions to study tertiary mathematics was administered by Forgasz (in press). Among the range of factors included were several affective variables. A slightly modified version of the questionnaire was used in the present study. The modifications allowed for a sample population of first year undergraduates and for a focus on mature-age students. Minor wording changes were necessary and a few new items tapping students' perceptions of their competence with calculators and computers were added. The items relevant to the findings reported in this paper included:

- Biographical and background information: sex, age, language background, and SES indicators. These formed the independent variables in the analyses.
- Beliefs about tertiary mathematics e.g., perceived usefulness, interest, and difficulty.

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[2] Students who are at least 21 years of age when they commence undergraduate tertiary studies

- Beliefs about self as a learner of mathematics e.g., enjoyment of mathematics, perceived achievement and competence levels, confidence of passing, and attributions for success and failure (to ability, effort, task, environment).

Students responded to most items measuring attitudes and beliefs on 5-point Likert type scales (1 = strongly disagree to 5 = strongly agree or 1 = weak to 5 = excellent). For the remaining items students selected among categories (e.g., yes/no/sometimes). Responses were analysed using SPSS_{WIN}. Four-way analyses of variance [ANOVAs] and chi-square tests were used to explore for differences and interaction effects among the various groups of students. Statistical significance was set at the .05 level.

Table 1. Sample sizes by grouping variables: gender, language background, age, socio-economic status [SES]

	Student Sex		Language		Age		SES	
	M	F	ESB	NESB	SL[2]	MA	Austudy	No Aus
Total N=781[1]	477	297	491	288	685	96	262	517
(%)	(62)	(38)	(63)	(37)	(88)	(12)	(34)	(66)
Males			302	174	412	65	166	310
Females			184	112	268	29	92	204
ESB					442	49	143	348
NESB					241	47	118	168
School leaver							213	471
Mature-age							49	46

[1] Seven students did not identify their sex, two did not indicate whether they were ESB/NESB, two did not indicate whether or not they received Austudy (See below for details of this grant)

[2] SL = School leavers; MA = Mature age

Context for the grouping variables used in the study

Unlike in the USA and the UK for example, where large minority groups are distinguished by racial/cultural backgrounds, Australia's multicultural profile is often characterised by language usage (e.g., the population census). Individuals can be considered to be of non-English speaking background [NESB] if a language other than English is frequently spoken in the home. This definition of NESB is one of a range of definitions that has been used in Australia over the years in a wide variety of data collections (see Yates & Leder, 1996) and was adopted in this study.

The Australian government provides financial assistance to some tertiary students through the Austudy scheme. For students up to the age of 25 to receive a level of financial assistance, combined parental incomes must be within a pre-defined range. The receipt of Austudy is therefore a measure of socio-economic status. Most commonly, students meeting tertiary requirements enter the sector directly from school. There are alternative entry paths for older students. While the definitions of and criteria for 'mature-age' entry vary slightly between institutions, commencing undergraduate students who are at least 21 years of age are generally categorised as 'mature-age'.

Results and discussion

On Table 2 the mean scores for the entire sample and by grouping variables (gender, age, language background, and SES) are shown for each variable for which at least one main effect in the 4-way ANOVA was statistically significant.

Table 2. Mean scores for affective measures by grouping variable

Variable	All	M	F	SL	MA	ESB	NESB	Austudy	No Aus
Self-ratings: perceived abilities									
Maths	3.54	3.58	3.48	**3.57**	**3.34***	3.56	3.51	3.48	3.58
Gr.12 maths	3.74	3.76	3.70	**3.77**	**3.49***	3.77	3.69	**3.55**	**3.83*****
Uni maths	3.37	3.41	3.30	3.35	3.49	3.33	3.44	3.32	3.39
Computer competence	3.64 (323)[1]	**3.89**	**3.30*****	3.64	3.67	**3.71**	**3.52***	3.78	3.57
Calculator competence	4.16 (514)	**4.26**	**4.01*****	4.19	3.98	**4.24**	**4.03****	4.20	4.14
Other affective variables (beliefs about self)									
S/Effort	4.11	**4.05**	**4.19***	**4.06**	**4.42****	4.05	4.21	4.16	4.08
F/Ability	2.85	2.79	2.94	2.84	2.93	**2.74**	**3.06*****	2.88	2.83
Confident of passing	3.73	**3.81**	**3.60****	3.73	3.76	3.74	3.70	3.68	3.75
Beliefs about tertiary mathematics (Mathematics at university is...)									
Challenging	3.93	**3.88**	**4.00***	**3.91**	**4.08***	3.95	3.89	3.93	3.92
Interesting	3.27	**3.22**	**3.35***	**3.22**	**3.65*****	3.20	3.39	3.38	3.22
Useless	2.21	2.19	2.24	**2.25**	**1.91****	2.21	2.22	2.17	2.23

NB. * $p < .05$ ** $p < .01$ *** $p < .001$

[1] Reduced sample sizes. Responses included were from students who responded 'yes' to: Are you expected to use computers/calulators in your mathematics course?

The data in Table 2 indicate the following differences among the various groups:

Gender differences (6). Compared to males, females:

- perceived themselves less competent in using computers and calculators
- believed that success was to due to effort to a greater extent
- were less confident of passing mathematics at university this year
- found tertiary mathematics more challenging and interesting

Differences due to age (6). Compared to school leavers, the mature age students:

- had lower self-ratings of their mathematical achievement levels and their levels of performance in mathematics at Grade 12.
- believed that success was due to effort to a greater extent
- found university mathematics more challenging, interesting and more useful

Differences by language background (3). Compared to those from English-speaking backgrounds, NESB students:

- perceived themselves less competent in using computers and calculators
- believed that failure was due to lack of ability to a greater extent

Socio-economic status differences (1). Compared to students from higher SES backgrounds (no Austudy), lower SES students:

- had lower self-ratings for their performance level in grade 12 mathematics.

These findings reveal that there were more intra-group differences for the independent variables 'gender' and 'age' than for SES and language background. To understand better the gender differences and those between school leavers and mature-age students, we also examined the four-way ANOVA results for two-way interaction effects with gender and for two-way interactions with age.

Two-way interactions with gender

For all except one of the variables which showed main effect gender differences, no two-way interactions were found. Thus the pattern of gender difference was essentially the same among students no matter how they were grouped. For 'university mathematics is interesting' and for three variables which had no main effect gender differences, there were statistically significant interaction effects. For 'self-rating of mathematics achievement' there were two interactions (language background and age). Overall, there were three two-way interactions with language background and one with age. These are listed below. Mean scores are shown.

- University mathematics is interesting: no differences for ESB males and females (M=3.20, F=3.21); NESB females scored higher than males (M=3.26, F=3.58).
- Self-rating of mathematics achievement: ESB males rated their achievements higher than females (M=3.83, F=3.66); the pattern was reversed for NESB males and females (M=3.64, F=3.76). Among the mature-age students, there was a larger difference in self-rating of mathematics achievement favouring males (M=3.48, F=3.00) than there was among the school leavers (M=3.60, F=3.53).
- Attribution of failure to lack of ability: ESB females scored higher than males (M=2.63, F=2.92); the pattern was reversed for NESB males and females (M=3.11, F=2.98).
- Self-rating of mathematics achievement at grade 12: ESB males rated their achievements higher than females (M=3.83, F=3.66); the pattern was reversed for NESB males and females (M=3.64, F=3.76).

These findings suggest that, in general, ESB males have more functional beliefs than females as do NESB females compared to males. For variables with main effect gender differences, the two-way interactions provide further insights into what other factors are contributing to them. While males and females may not differ on particular affective measures, particular subgroups within the population do exhibit gender differences. Previous research frequently reports gender differences favouring males on variables such as 'self-rating of mathematics achievement'. Were it not for the very high-scoring NESB females, this usual pattern would be evident in the data gathered in the present study. It would be very convenient to conclude that there were no gender differences on this variable. However, the explorations of other contributing social factors have revealed patterns worthy of further investigation.

Two-way interactions by age

For three of the six variables showing main effect differences by 'age' (beliefs about tertiary mathematics being challenging, interesting and useless), and for all but one of the five variables for which no statistically significant main effect differences were found, there were no two-way interaction effects. The patterns of difference or no difference between school leavers and mature-age students were thus the same for each of the other grouping variables.

There were four statistically significant two-way interactions with 'age'. The two-way interaction for 'self-rating of mathematics achievement' with gender was discussed above. Of the remaining two-way interactions, two were with language background; the third was with SES. Mean scores are given:

- Self-rating of mathematics achievement at grade12: very little difference between the self-ratings of NESB mature age students and school leavers (SL=3.69, MA=3.72); ESB school leavers rated their achievements higher than mature-age students (SL=3.82, MA=3.29).
- Success attributed to effort: very little difference between NESB mature-age students and school leavers (SL=4.19, MA=4.16) in attributing success to effort; ESB mature-age students scored higher than school leavers (SL=4.00, MA=4.63).
- Confidence in passing mathematics at university (no main effect difference by age): Among Austudy recipients (lower SES), schools leavers were more confident than mature-age students (SL=3.95, MA=3.59); the pattern was reversed for those not receiving Austudy (SL=3.63, MA=3.77).

Language background was also an important variable, illuminating differences among mature-age students and school leavers from different language backgrounds. It was the ESB students who contributed to the main effect differences by age.

In summary, language background appeared to provide partial explanations for patterns of gender difference (and sometimes as explanations for no gender differences overall) and for patterns of difference by 'age'. Language background appears to be a critical variable worthy of inclusion in studies exploring beliefs and attitudes about mathematics at the tertiary level in Australia. In other countries, equivalent variables might be race or cultural/ethnic background. SES did not appear to have as large an impact on the gender and age-related differences evident in the sample surveyed.

Table 3 contains percentage frequency distributions and results of the chi-square tests on the items tapping enjoyment of mathematics at school and at university.

Table 3 Percentages of students who enjoyed mathematics by grouping variable

	%	M	F	SL	MA	ESB	NESB	Austudy	No Aus
Enjoyed	Yes	57	57	58	53	57	57	58	57
Maths at	No	11	8	9	20	10	9	10	10
School	S'times	32	35	33	31	33	34	32	33
Enjoy	Yes	**37**	**37***	**35**	**48***	34	**41****	**42**	**33****
Maths	No	**22**	**14**	**19**	**15**	**23**	**12**	**21**	**18**
At uni.	S'times	**42**	**49**	**46**	**37**	**43**	**47**	**37**	**49**

Significant differences in response distributions: $^{*}p<.05$ $^{**}p<.01$ $^{***}p<.001$

Table 3 reveals a general decline in the enjoyment of mathematics between school and university. There were no significant differences in the distributions of responses for each grouping variable with respect to the enjoyment of school mathematics. However, with respect to tertiary mathematics, differences emerged for each group:

- males were more inclined not to enjoy university mathematics than females
- mature-age students, NESBs and Austudy recipients enjoyed university mathematics more than their respective counterparts

The rapid decline in students' enjoyment of mathematics from school to their first year of tertiary mathematics study is noteworthy. Why this has occurred should be of concern to tertiary mathematics teachers. Could it be that there are differences in the school and tertiary teaching/learning environments that have resulted in this change? The finding that 'disadvantaged' groups – those of non-English speaking backgrounds, older students, and those from lower socio-economic backgrounds – are less affected by the transition is also worthy of further investigation. In having reached the higher education sector, do these students have greater motivation to achieve in their tertiary studies? This finding also re-inforces the contention that factors other than gender alone are crucial in understanding differences in participation rates in tertiary mathematics.

Final words

Gender differences in mathematics learning have attracted much research attention. The findings of the present study confirm that an examination of more subtle within-group differences yields a fuller understanding of factors influencing participation and performance in tertiary mathematics. Self-responses from a representative sample of first year university students revealed not only a number of gender differences, but also differences between school leavers and their older 'classmates' (i.e., mature age students) in self-perceptions and responses to aspects of the mathematics subject(s) in which they were enrolled and between students from different home backgrounds. For example, ESB males appeared to have more functional beliefs about mathematics and their ability to cope with the subject than ESB females; but NESB males had less functional beliefs than their female counterparts. ESB school leavers rated their year 12 mathematics achievements, and the role played by effort in attaining success, higher than mature age ESB students. No such differences were found for students from an NESB background. NESB students, those receiving Austudy, and students designated as mature age, seemed to enjoy tertiary mathematics more than, respectively, ESB students, those from higher socio-economic backgrounds, and students moving directly from school to university. While important in themselves, the survey findings reported in this paper can fruitfully be supplemented with more detailed information gathered through case studies and interviews. These comprise the next stage of our research.

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Social Class Inequalities in Mathematics Achievement: A Multilevel Analysis of "TIMSS" South Africa Data.

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This paper describes the extent of inequalities in South Africa's grade 8 mathematics classrooms. Inequalities in mathematics achievement associated with social class differences were found to be significant and vary from class to class. There are significant sex differences favouring males and the difference is especially large in high achieving classes. Achievement gaps also exist between classes because of the social class background of the students in the class. An achievement gap of more than half a standard deviation exists between high and low SES schools. The general attitude of students in a class towards mathematics, rather than the individual students attitudes towards mathematics, is more important in reducing social class inequalities and levels of mathematics achievement between classes.

Research has increased our understanding of how schools and classrooms affect children from diverse backgrounds by addressing two major questions: (1) To what extent do schools or classrooms vary in their outcomes for students of differing status? and (2) What school and classroom practices improve levels of schooling outcomes and reduce inequalities between high and low-status groups? (Willms, 1992). The term "social-class gradient" refers to the relationship between an individuals' educational outcomes and their socioeconomic status (SES). SES is a description of an individual's or a family's social class. Factors such as income, the prestige of a person's occupation, and their level of education are usually used to measure SES (White, 1982). Gradient can also be a measure of the inequality in an educational outcome between males and females.

Research shows that there is a relationship between SES and school outcomes especially academic achievement. White (1982) reviewed research on SES and achievement using meta analysis. The average of about 489 individual- level correlations he analysed was 0.25 with a standard deviation of 0.16 in the correlations across studies. In mathematics, the mean correlation was 0.20.

Researchers have often used Bourdieu's (1977) "cultural capital" theory to explain the positive relationship between SES and school outcomes. Cultural capital is about the values, norms of communication, and organizational patterns possessed by middle class parents. The argument is that schools are largely controlled by the middle class who impose their values, language patterns and organizational patterns on schools (Lamont & Lareau, 1987). Middle class parents find it easier to relate to school authorities and are frequently involved in school activities with the sole objective of achieving what is best for their children (Lareau, 1987). Children raised in middle class environments, through the participation of these language patterns at home, and the expectations from parents, possess the cultural capital that enable them to adapt and fit easily into the school environments.

However, school environments and classrooms differ in terms of composition of students' population. Although school population depends on a number of factors including, the geographical location of a school, parental interest, students' preferences, or admission policies including some measure of the students' aptitude, social class play a major role in the social composition of the school population. In simple cases, a school population will reflect the social class composition of the neighbourhood it serves while, in complex cases where the attendance area embraces different neighbourhoods, the school population will be more heterogeneous, but usually a particular social class will dominate. Schools

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are therefore segregated in terms of social class.

Research indicates that the social background characteristics of students in a school have substantial effect on students school outcome over and above the students own ability and social class (Willms, 1992). This is what is referred to as "contextual effect". School or classrooms with students from high social-class parents tend to have several advantages associated with the learning contex. These schools are also more likely to attract and retain talented and motivated students. The result of segregation of schools along social class lines is that students from social class advantage background not only do better but are also more motivated to learn than those from disadvantage backgrounds.

Research shows that in general students are motivated to learn mathematics and do well when they have a positive attitude towards mathematics (ATM). The review of research on ATM and mathematics achievement indicated that most of the studies have been done at the student level with virtually no study in the school or classroom level to determine for instance how the "general" ATM of students' in a particular class affects their mathematics achievement over and above their own attitude towards mathematics and their social class backgrounds (see review in Ma & Kishor, 1996). The main objective of this study is to assess the extent of inequalities within and between grade 8 classrooms in South Africa, especially, with respect to social class. The effects of students' ATM and the general ATM of a class on mathematics achievement and inequalities in mathematics achievement are also assessed.

The South Africa grade 8 data from the Third International Mathematics and

Science Study (TIMSS) are used in this study. This study was conducted under the auspices of IEA (International Association for the Evaluation of Educational Assessment) in 1994-95 academic year in more than 40 countries. In the population 2 study, the mathematic achievement test was administered to a selected grades 7 and 8 students from a randomly sampled schools within the countries. Students and teachers also responded to a set of questionnaire about themselves, their schools and classrooms.

Multilevel statistical models were employed in this study. Over the past decade, the development of multilevel statistical models has resulted in the conceptualization and estimation of models pertaining to levels of outcomes and gradients for data that have a nested structure, such as students within a classroom (Bryk & Raudenbush, 1992). In classroom effects research, the approach will entail the estimation of a separate regression equation for each class, which will yield a set of intercepts (i.e., levels of outcome for each class) and slopes (i.e., gradients for each class). The set of intercepts and slopes become the outcome variables at the second level of the model, which can be regressed on variables describing classroom processes.

Models and Data Analyses.

The analyses of the data involved 5 models. The first model (model 1) usually called the "null" model does not include any independent variables. This model simply partitions the total students' variation on the dependent variable (mathematics achievement score) into within class and between class variations (similar to what is done in Analysis of Variance) - the mathematics achievement score is standardized. In the second model, I included the SES measure into the

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equation to determine the extent of the effect of SES on mathematics achievement. The educational level (measured in years of education) of mother and father of students along with some 16 selected items, including books, TV and a car in a student's home were aggregated and scaled to have a mean of 0 and standard deviation of 1. Sex and ATM were added into the equation in Model 3 to determine the effect of these variable on SES and also on mathematics achievement. MeanSES was added to the Model 4 to assess the contextual effects on levels of mathematics achievement and on SES and sex inequalities. MeanATM entered the equation in Model 5 to determine the independent effect of the ATM of a class on the levels of mathematics achievement, and on SES and sex gradients (See Table 1 for the description of these variables).

Table 1

Means, Standard Deviations, and Descriptions of Variables

Variable	Mean	SD	Description
Student-level variables (N = 4491)			
Sex	0.51	0.5	Coded as: Male=0, Female=1.
SES	0	1	Socioeconomic status.
Attitude Towards Mathematics (ATM)	2.87	0.997	Scale coded as dislike a lot =1, dislike=2, like=3, like a lot =4.
Classroom-level variables (N=116)			
MeanSES	-0.02	0.59	Mean SES of a class
MeanATM	2.85	0.339	Mean ATM of a class

Table 2

Hierarchical Regression Coeffecients, Variances and Correlations.

	Models	model 1	model 2	model 3	model 4	model 5
Student-level Equation		Regression Coeffecients				
Constant			-.119**	-.120**	-.116**	-.108**
SES			.037*	0.036	0.018	0.021
Sex				-.069**	-.073**	-.074**
ATM				.070**	.072**	.065**
Class-level						
SES	MeanSES				.176**	.165**
	MeanATM					.122**
Sex	MeanSES				-.074	-.060
	MeanATM					-.082
Adjusted Score	MeanSES				.595**	.543**
	MeanATM					.480**
Variation						
Within Classes		0.541	0.539	0.534	0.533	0.534
Between Classes	Average Score	.382**	.223**	.208**	.097**	.073**
	SES Gradient		.020*	.021*	.007*	.005*
	Sex Differences			.006	.005	.003

Correlation Between Parameter Variances for model 5			
	1	2	3
Adjusted Class Score (1)	1.00	-0.82	0.86
Sex Differences (2)	-0.82	1.00	-0.97
SES Gradients (3)	0.86	-0.97	1.00

The estimates from the five models are displayed in Table 2. Model 1 indicates that about 58.6% of the total variation on students' mathematics achievement is within classes (.541/(.541+.382)) and 41.4% is between classes. The between class variation was significant ($p<.001$) suggesting that the levels of mathematics achievement differ from class to class.

When the SES variable was added to the equation it reduced the between class variation by about 41.6% (from .382 to .223). The effect of SES on mathematics achievement was .037 which was significant at $p<.05$. This is small compared to what pertains in the literature. However, the variation on SES gradients (.020) was significant at $p<.01$ suggesting that classes differ on their SES gradients. With a variance of .02 (S.D.=.141) the SES effect could be as high as .330 (.141*2 + .037) in some classes. The SES gradient is positively correlated with level of class achievement which means that the social class inequality is wider in schools with high levels of mathematics achievement.

In model 3 when sex and ATM were added to the equation the variation in adjusted class achievement scores reduced by only 6.7% (from .223 to .208). The sex effect (adjusted for SES and ATM effects) was significant at $p<.01$. The difference in achievement between the sexes was about .069 in favour of males. The variance of .006 on the sex gradients was not significant. There is a negative correlation between the sex gradients and class achievement levels suggesting that in classes where there is a high achievement, sex differences favour males. The ATM effect (.070) was significant but did not seem to have made any significant impact on either the SES gradient - the SES effect reduced from .037 to .036 when sex and ATM were added to the model.

In models 4 MeanSES reduced the variance by 53.4% (.208 to .097) and between class SES gradients by 66.7% (.021 to .007). This means that much of achievement differences between classes are mainly due to contextual effects - the effect of the class SES on students achievement irrespective of their social class background and attitude towards mathematics. The meanSES effects on SES inequalities and adjusted class achievement levels were both significant at $p<.01$. For two schools with a unit differences (one standard deviation) in meanSES, the achievement gap is about .176 between low and high SES students and there is an achievement difference of about .595 between the classes. The MeanSES flattened the SES gradient by about 50% (from .036 to .018) but did not appear to have any effect on sex inequalities. The meanATM had significant effect on meanSES inequalities and adjusted class achievement levels. For two schools with about a unit difference in meanATM, there is an achievement gap of about .12 between high and low SES students in favour of high SES students and an a gap of about .480 between the two schools. A change in attitude of students in a class from disliking to liking mathematics is likely to increase the average mathematics achievement of that class by 48% of a standard deviation. MeanATM also reduced the variance on the SES gradient by about 28.6% and between class variance by 24.7%

Conclusion

These analyses suggest that much of the differences in class achievement is associated with the social class background of students and the composition of students in terms of their background characteristics. There is an achievement gap of about 4% of a standard deviation between students with high SES families and those from low SES families in favour of those from high SES families. In high achieving schools, female students and students from low SES families tend to perform poorly in mathematics. The mathematics achievement gap between high SES schools and low SES schools is about 60% of a standard deviation. The general students' attitude towards mathematics in a class rather than individual student's attitude will be more effective in reducing mathematics achievement gaps between schools and the gap associated with the background characteristics of students.

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CONTEXT INFLUENCE ON MATHEMATICAL REASONING

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ABSTRACT. *The problem studied in this paper is how and under which conditions students accept or refuse the rules of formal deduction. In particular, the focus is on the role of the context in the activity of proving, where by 'context' we mean the 'semantic context' of the statement to be proved and not the global context in which the classroom is set. Our study is based on the analysis of the answers of 40 students aged 16 years to a questionnaire on the introduction and elimination of 'and', and on the introduction of 'or'. The results of our analysis reveal, in our opinion, a remarkable interference of the context, which includes both the semantic meaning of the propositions involved in a deduction step and certain implicit assumptions induced by the common usage of certain words in the natural language; this is particularly evident in the case of the introduction of or.*

1. INTRODUCTION AND OUTLINE

BACKGROUND. In the past the traditional theories about human reasoning developed in the field of psychology used to assume that a mental 'natural' logic exists structured around the pattern of classic formal logic. We can observe a consequence of this assumption in school practice: students are asked to accept some rules as self-evident, even though they feel these rules are not. That position is questioned by recent empirical studies, which show that human performances in reasoning are far from the ideal correctness of classic formal logic. For example, the papers (Balacheff, 1988; Chazan, 1993; Martin & Harel, 1989; Porteous, 1991) show that students prefer empirical rather than hypothetical-deductive reasoning. Other studies, for example (Zazkis, 1995), point out the use made by students of alternative schemes of reasoning. It is important to note that this issue is central in the studies on artificial intelligence.

Duval (1991 and 1992-1993) has made the important point that usual reasoning (i.e. the one performed, in a variety of contexts, rather informally in a natural language) and formal reasoning (formal deduction), though apparently similar in some respects, are actually deeply different. Indeed, as Duval rightly argues, in usual reasoning the rules of inferential deduction are implicit and the conclusions strongly depend on the semantic meaning of the premises, while in formal reasoning the inferential rules are explicit and the conclusions are deduced independently of the semantic meaning of the premises (in Duval's words, only on the basis of the "operational status" of the propositions involved).

In a previous research, see (Furinghetti & Paola, 1996), we have investigated some aspects of the above issue, i.e. how and under which conditions students accept or refuse the rules of formal deduction, through a questionnaire and interviews. From our findings we argued that:

- quite often the rules of formal deduction, implicitly used in classroom mathematical practice, are perceived as unnatural compared to usual reasoning
- when engaged in some deductive reasoning, almost invariably the students try to associate a meaning with the premises and are strongly influenced not only by that meaning, but also by the overall context allowing to give a meaning to the propositions involved. We can say that in certain cases the semantics provided by the context may act as an element of diversion in proving.

Since the general information provided by that work needed to be checked and refined, we have gone further with the research reported here, which is aimed at investigating how students deal with the rules of:

- the introduction of the conjunction *and*, formally

$$\frac{H \quad K}{H \wedge K} \quad \text{(i.e. from } \{H, K\} \text{ we can infer } \{H \textit{ and } K\}\text{)}$$

- the elimination of the conjunction *and*, formally

$$\frac{H \wedge K}{H} \quad \text{(i.e. from } \{H \textit{ and } K\}\text{) we can infer } H\text{)}$$

together with the analogous obtained by exchanging H for K and conversely

- the introduction of the disjunction *or*, formally

$$\frac{H}{H \vee K} \quad \text{(i.e. from } \{H\} \text{ we can infer } \{H \textit{ or } K\}\text{)}$$

together with the analogous obtained by exchanging H for K and conversely.

Our main emphasis is on the influence of the context on the application of such rules, since our working hypothesis is that, whenever the association of meaning to propositions is impossible or beyond the students' ability, then we may experience either a student being stuck or, paradoxically, giving a better performance than within a context providing too many information (i.e. less answers, but, proportionally, a better percentage of correct answers). This is not opposed to, but rather complementary to the findings of some studies, like for example (Chazan, 1993), where it is shown that an appropriate semantic context may be of help in proving.

The research here reported is part of a large project on mathematical reasoning, under development since a few years and investigating different aspects of that activity, like mastering the mathematical language, role of intuition and of rigour, visualization, empirical checking of conjectures or statements, impact of computers.

METHODOLOGY. We have assumed that dealing with the rules of deduction requires a certain mathematical experience. For this reason the present study has addressed students aged 16; indeed, according to our national curricula, they have already been trained in proving some theorems (mainly in arithmetic and geometry). The research has been performed by means of a questionnaire to be answered in half an hour, assigned to 40 students of two classes of an Italian high school. In the curriculum of this school mathematics has an important role. The questionnaire is three pages long; it contains 12 questions, with 4 options plus the possibility of a comment. The questions are grouped in 3 sections (**i1**, **i2**, **ii**): the first two concern the conjunction *and*, the

third the disjunction *or*. Each section consists of 4 questions constructed according to the same schema, but set in the following different contexts:
Context **A**. Mathematical context rich of meaning for the students (analytical geometry, which constitutes the main part of the mathematical program)
Context **B**. Mathematical context without meaning for the students (metatheoretical results of mathematical logic)
Context **C**. Everyday experience rich in meaning (card-game)
Context **D**. Artificial context (the sentences in the questions are expressed in the natural language, in a grammatically correct form, to which we have added three imaginary words; thus the sentences are without any meaning but those the students themselves can attribute).

After having collected the data we have proposed the same questionnaire to 9 students of the fourth year of the mathematics course at the University. They went through a great amount of theorems in analysis, geometry, topology. We do not discuss in detail the results of this last category of students; we only point out that the trend of their answers is analogous to that of the younger students. The interest of the further information provided by these university students is in the comments they write. While the younger students showed a poor ability in explaining their strategies or why their are not able to answer, the university students showed a certain awareness of what they do and why they do it. Thus their comments are clearer and richer in information than those of their younger colleagues.

2. EVALUATION OF THE EXPERIMENTAL RESULTS

Section II (*Introduction of the connective 'and'*)

In all the 4 questions of the section the options are as follows: [*a*] nothing, [*b*] the right answer, [*c*] manifestly wrong, [*d*] two right propositions, but the second is not deducible (in terms of formal rules) from the two given propositions. In the tables for each question we report the percentages of right answers; the data not reported refer to null percentages of answers.

Question IIA. Given the two following propositions:
1. *The straight line of equation* $x - 2y = 0$ *passes through the point* $(2,1)$
2. *All the straight lines non parallel to the straight line of equation* $x - 2y = 0$ *passing through* $O(0,0)$ *have in common with the straight line of equation* $x - 2y = 0$ *the point* $O(0,0)$

Using the rules that you habitually use when proving what you can conclude from the two propositions (only from them)?
[*a*] nothing
[*b*] *The straight line of equation* $x - 2y = 0$ *passes through the point* $(2,1)$ and *All the straight lines non parallel to the straight line of equation* $x - 2y = 0$ *passing through* $O(0,0)$ *have in common with the straight line of equation* $x - 2y = 0$ *the point* $O(0,0)$
[*c*] *The straight line of equation* $x - 2y = 0$ *passes through the point* $(2,1)$ and *The straight line of equation* $x - 2y = 0$ *passes through the point* $(1,0)$
[*d*] *The straight line of equation* $x - 2y = 0$ *passes through the point* $(2,1)$ and *The straight line of equation* $x - 2y = 0$ *passes through the point* $(4,2)$

[*] Brief comment (optional)

a	*b*	*c*	*d*	*a,d*	*b,d*	*b,c,d*	*only comment*	*no answer*
0	27.5	0	18	2.5	50	2.5	0	0

The semantic context is familiar to the students, since they are studying analytical geometry at the moment they answer the questionnaire. They do not feel the need of adding any comments, as we will see it happens when the semantic context is not well mastered. The main difficulty is to consider *only* the given propositions, since students tend to give as much information as they can. In this case the second proposition in the option [*d*] is some additional information easily obtained by means of calculations in the field of analytical geometry. Performing these calculations is, in the students' mind, what the teacher is expecting from them. This fact may explain the preference accorded to [*d*]. Another cause of this preference can be found in the students' reluctance to draw conclusions which are trivially contained in the premises, as it is in the case of the option [*b*]. It may be interesting to note that a student who answers correctly all questions of Section **i1**, underlines the words "only from them" in the text of the questionnaire.

Question i1B. Given the two following propositions:
1. *PA is incomplete*
2. *All the extensions w consistent recursively axiomatized of PA are incomplete*

Using the rules that you habitually use when proving what you can conclude from the two propositions (only from them)?
[a] nothing
[b] *PA is incomplete* and *All the extensions w consistent recursively axiomatized of PA are incomplete*
[c] *PA is incomplete* and *The number of integers is finite*
[d] *PA is incomplete* and *The number of odd numbers is infinite*
[*] Brief comment (optional)

a	*b*	*c*	*d*	*a,b*	*only comment*	*no answer*
10	55	0	0	5	25	5

The percentage of right answers is considerably increased. In this totally unknown context there is no misleading information due to the interference of the experience (from school, as it was in the case of analytical geometry or from outside school, as we will see in the case of the card-game). The students have less information to organize and have the possibility to be more concentrated on the rules of deduction. On the other hand we observe that the lack of semantic context raises disorientation, as shown by the increasing number of students answering that nothing can be concluded and by the number of comments which are given not only in isolation, but also accompanying the various answers; for example, 40% of the right answers are accompanied by a comment. The student who in the question **i1A** has underlined the words "only from them" answers rightly, but adds "What does the sentence mean?". Another one, also providing good answers, points out "I have chosen the option [*b*] by logic, but I do not know what does it mean". The most common comment is of the type "I do not know what the words recursively and axiomatized mean and thus I am not able to give a sensible answer to this kind of questions".

Question i1C. Given the two following propositions:

1. *The playing-cards are of two colours*
2. *All the red cards not diamond are heart*

Using the rules that you habitually use when proving what you can conclude from the two propositions (only from them)?

[a] nothing

[b] *The playing-cards are of two colours* and *All the red cards not diamond are heart*

[c] The *playing-cards are of two colours* and *The spade ace is red*

[d] *The playing-cards are of two colours* and *The club ace is black*

[*] Brief comment (optional)

a	*b*	*c*	*d*	*b,c*	*b,d*	*only comment*	*no answer*
2.5	35	0	2.5	2.5	52.5	2.5	2.5

We see that the trend is similar to that of the question **i1A**. In both cases for the majority of students we find that the richness of information contributes to hide the formal structure of the text and the deductive step is not made thanks to the formal role of the propositions, but thanks to their meaning. These students accept the introduction of propositions that are true, even when they are not those given at the beginning. However in both cases all students feel confident and do not feel the need to add comments, apart one who asks about the number of the cards and the colours.

Question i1D. Given the two following propositions:

1. *Lim is a tohgh*
2. *All the pohl rhythmic consulted of a Lim are dusty*

Using the rules that you habitually use when proving what you can conclude from the two propositions (only from them)?

[a] nothing

[b] *Lim is a tohgh* and *All the pohl rhythmic consulted of a Lim are dusty*

[c] *Lim is a tohgh* and *The men are immortal*

[d] *Lim is a tohgh* and *The men are mortal*

[*] Brief comment (optional)

a	*b*	*c*	*d*	*a,b*	*b,d*	*only comment*	*no answer*
7.5	52.5	0	2.5	5	2.5	27.5	2.5

In the above propositions three words ('Lim', 'pohl', 'tohgh') occur, which do not exist in the Italian language (nor in the English language too). The numerical data show a trend similar to that of the question 2. Even in this case the lack of semantic context provokes disorientation, as witnessed by the high percentage of comments, not only in isolation, but also accompanying the various answers. However, the lack of meaning does not introduce elements of diversion, as indicated by the low percentages of answers containing the option [*d*].

Section i2 (*Elimination of the connective 'and'*)

In all the 4 questions of the section the options are as follows: [*a*] nothing, [*b*] right answer, [*c*] right answer, [*d*] a proposition true, but not deducible (in terms of formal rules) from the two given propositions. To give an idea of the kind of questions of this section we report the text of the second question **i2B**, whose context is mathematical logic. In the four tables after the question the results are presented with the same criterion used in Section **i1**.

Question i2B. Given the following proposition:
PA is incomplete and All the extensions w consistent recursively axiomatized of PA are incomplete
Using the rules that you habitually use when proving what you can conclude from the two propositions (only from them)?
[a] nothing
[b] *PA is incomplete*
[c] *All the extensions w consistent recursively axiomatized of PA are incomplete*
[d] *PA is incomplete* and *The number of odd numbers is infinite*
[*] Brief comment (optional)

Context **A** (Analytical geometry)

a	*b*	*c*	*d*	*b,c*	*b,d*	*b,c,d*	*only comment*	*no answer*
0	0	10	12.5	42.5	2.5	27.5	0	5

Context **B** (Mathematical logic)

a	*b*	*c*	*d*	*b,c*	*a,c*	*a,b,c*	*only comment*	*no answer*
2.5	5	5	0	52.5	2.5	5	17.5	10

Context **C** (Cards game)

a	*b*	*c*	*d*	*b,c*	*b,d*	*c,d*	*b,c,d*	*a,b,c,d*	*only comment*	*no answer*
0	2.5	5	2.5	15	2.5	7.5	52.5	2.5	2.5	5

Context **D** (Natural language without meaning)

a	*b*	*c*	*d*	*b,c*	*a,b,c*	*b,c,d*	*a,b,c,d*	*only comment*	*no answer*
7.5	0	7.5	2.5	45	5	5	2.5	17.5	7.5

The percentages show a trend rather similar to that of the first part of this section. In the contexts rich of meaning, **A** and **C**, the option [*d*] containing right sentences is chosen by 42.5% and 67.5% respectively versus the 0 and 2.5 in the contexts **B** and **D**, both as for the influence of the context and as for a rather good level of acceptance of the rule (here the elimination of *and*). The comments of the University students point out a possible specific reason of this preference. For example, a student writes that she cannot deduce anything since the propositions give information she already has. This student considers deduction as an activity which necessarily has to increase the amount of information. Another student writes "I cannot deduce anything, but what I already know".

Section ii (*Introduction of the connective 'or'*)

In all the 4 questions of the section the options are as follows: [*a*] nothing, [*b*] right answer, [*c*] right answer, [*d*] right answer. To give an idea of the kind of questions of this section we report the text of the third question **iiC**. In the four tables after the question the results are presented with the same criterion used in the Section **i1**.

Question iiC. Given the two propositions:
1. *The playing-cards are of two colours*

2. *All the red cards not diamond are heart*

Using the rules that you habitually use when proving what you can conclude from the two propositions (only from them)?

[a] nothing

[b] *The playing-cards are of two colours* or *All the red cards not diamond are heart*

[c] *The playing-cards are of two colours* or *The spade ace is red*

[d] *The playing-cards are of two colours* or *The heart ace is black*

[*] Brief comment (optional)

Context **A** (Natural language without meaning)

a	*b*	*c*	*d*	*b,c*	*b,d*	*c,d*	*b,c,d*	*only comment*	*no answer*
10	27.5	2.5	10	2.5	10	2.5	7.5	17.5	10

Context **B** (Analytical geometry)

a	*b*	*c*	*d*	*a,c*	*c,d*	*b,c,d*	*only comment*	*no answer*
22.5	27.5	0	2.5	2.5	2.5	2.5	25	15

Context **C** (Analytical geometry)

a	*b*	*c*	*d*	*a,c*	*b,d*	*b,c,d*	*only comment*	*no answer*
15	7.5	2.5	7.5	2.5	32.5	2.5	0	5

Context **D** (Analytical geometry)

a	*b*	*c*	*d*	*a,c*	*b,d*	*b,c,d*	*only comment*	*no answer*
12.5	20	5	5	2.5	5	2.5	32.5	15

To emphasize the students' difficulties in the use of the disjunction *or* we show altogether the data about the right answers in the following table.

Question iiA	*Question iiB*	*Question iiC*	*Question iiD*
7.5%	0%	2.5%	0%

The analysis of the answers points out that the meaning given to *or* is mainly exclusive (*aut*). This fact comes from the natural language in which *or* is mainly used in this way, e. g. "do you want tea or coffee?" means that you are supposed to have *only* one type of drink. We point out that, while the younger students in their comments write generic sentences such as "it does not make sense", "it is impossible to answer", the University students offer precise explanations of the incorrect answers. They explicitly write "I have interpreted *or* as *aut*"; "it is wrong to use *or*, since both statements are deducible from the propositions and one does not exclude the other".

In (Johnson-Laird, 1993) there is an interesting explanation of the cause of the difficulty in the use of the connective *or*: it is ascribed to the non acceptance by the students of 'semantic dissipation'. For students to deduce $H \vee K$ from H is to dissipate meaning. For example, it is observed the reluctance in accepting that from the proposition "for any real x, $x^2 + 1$ is greater than 0" it is possible to deduce the proposition "for any real x, $x^2 + 1$ is greater or equal to 0". This deduction is considered "wrong" or "without sense", or "useless and thus to be refused".

Apropos of this fact Freudenthal has observed that students are used to tell not only 'the truth' but 'all the truth'. By the way this attitude is also suggested by the common sense in the situations of everyday life. Thus we have that both natural language and common sense generate conflicts in the use of *or*.

CONCLUSIONS

In the paper (Furinghetti & Paola, 1997) we have shown that the performances in proving are not 'context free', because of the many elements which may interfere negatively in the students' behaviour. In that case the interference came from arithmetic and algebra. To have focussed on the specific rules concerning *and*, *or* has allowed us to identify other interferences, those originated by the common sense, the natural language, the richness of information provided by the context. In this kind of interferences we can find one of the causes of the cognitive rupture in passing from the phase of construction to the phase of formal settlement of a proof. Our findings support the idea that this passage is not 'natural'; even more, that there are moments in which the two phases are opposing each other, for example, when the syntactic aspects of formal proof are in conflict with certain aspects of the usual reasoning. This fact induces us to think that, in order to introduce the pupils to the proving process, it is necessary to propose tasks leading them to perceive and to use the propositions on the basis of their operative status within the inference rule and not of their semantic meaning. To succeed it is necessary that students' attention be concentrated on the control of the deductive organization, giving the due relevance to the inferential rules.

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What Do They Really Think?

What Students Think About the Median and Bisector of an Angle in the Triangle, What They Say and What Their Teachers Know About It.

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The subject of special segments in the triangle raises some specific difficulties. We use it as an example of a process of identifying difficulties during learning, and observing the way teachers deals with them. The purpose is to draw teachers' attention to the possibility that some difficulty is hidden (or even not- hidden) behind the students' answers, to introduce an opening towards understanding the difficulties, and to increase their motivation to look for solutions. This is part of a broader research concerning identifying difficulties in Geometry instruction, analyzing them according to cognitive theories, and suggesting the analysis to the teachers in order to help their decision making during instruction (according to the paradigm of the Cognitively Guided Instruction (Carpenter & Fennema, 92).

Background and Method

The comprehensive research (see also *Gal & Vinner, 97*) tries to identify difficulties in geometry, analyze and find their sources, and suggest this pedagogic knowledge to the teachers in order to take it into consideration during their instructional decision making. This is a qualitative research which involves 7 pre-service teachers, in their third and fourth year of study. (Concerning the type of difficulties detected, the researcher does not distinguish between pre- and in- service teachers. Hence, from here on, the term "teacher" would describe pre-service teachers as well). They were teaching geometry to 8th-9th graders (slow and fair learners), 2 hours a week throughout the year. The lessons were partly videotaped, the teachers were interviewed before and mainly after the lessons and their lesson outlines were read.

In this paper we demonstrate some stages with two teachers, Alon and Udi, pre-service teachers in their third year of study. Alon and Udi taught geometry in the same class, alternately, both present in the classroom. There were about 18 students in class, mainly slow learners. The class was studying the subject of special segments in a triangle, and our interest was mainly bisector of an angle and median. (The median was the first to be taught). The article describes a four stage process: a. Observing a lesson about the bisector and identifying difficulties. b. Discussion with the teachers about the lesson. c. Observing the lesson that follows the interview. d. Interview and conversation with the teachers.

a. A Lesson About The Bisector of an Angle in a Triangle

The lesson was led by Alon. He was demonstrating how to find the bisector by using paperfolding, (mainly it was he who performed the folding...), he pointed out the differences between a bisector of an angle and a bisector of an angle in a triangle and defined each of them. Then, the students were given worksheets on which different triangles where drawn, and they were asked to draw the bisector.

The observations show that many of the students did not pay attention to the bisecting of the angle: No controlled procedure was observed. For example, a student used a wide non-transparent ruler, in order to draw the segment from the vertex to the opposite side. But since the angle and the opposite side were partly hidden behind the ruler the student could not estimate "where is the middle". It did not bother him...

In many cases the students bisected the side opposite the angle (instead of bisecting the angle itself). It could be seen in several ways: First, the segment that they drew in the triangle did not bisect the angle but did *bisect the opposite side.* Second, in some cases, the students were *"searching" for the middle of the side* in different ways: measuring with the ruler, "measuring" with their finger, etc. Third, in some other cases, students *started their drawing from the middle of the side* in the direction of the vertex, while in drawing the bisector of an angle the opposite is expected: start from the vertex.

We may conclude that part of the students *"bisected the angle"* by drawing segments from the vertex to the opposite side and from the vertex to the middle of the opposite side, regardless of the attribute of the segment as the bisector of the angle!

No reaction of the teachers was noticed to any of these cases. Furthermore, the teacher did not explicitly tell the students how to draw the bisector. Probably, he did not find a special difficulty in drawing a bisector of an angle and considered it (rather unconsciously) as a simple application of the previous part of the lesson.

b. Discussion with the teachers about the lesson

After the lesson the teachers were interviewed and the difficulties were discussed:

1. While planning the lesson, Alon did not expect any special difficulties in the assignment of drawing bisectors of an angle in the triangle.
2. When Alon hears about the "median instead of bisector" seen by the interviewer, he explains it by confusion with the other special segments.
3. Alon has a *relational* instructional motivation. (Interviewer*: For what purpose do we teach the bisector of an angle?* Alon:*...It's because of the mathematical meaning of this issue).* He does not specify what is the mathematical meaning that he finds in the bisector of an angle, but still there is no clue in his words to instrumental goals of studying this issue. On the contrary! He wants the students to *understand!* (Alon: *In fact, I want them to understand what the bisector of an angle does in a triangle*).
4. When a student declared that he was drawing a bisector of an angle, though it did not look like it, the teacher refered to the answer as imprecision. (Alon: *When they say that they drew a bisector though I see that it's not equal, but they say: "I meant it to be equal, it just didn't work out", then I accept it as O.K.).* Alon does not treat such an answer as misconception or as a different purpose in drawing the segment. The teacher does not take into consideration the possibility of a gap between the *verbal declarations* and the student's *concept image* (*Vinner, 91).*

I would like to specify this point. The student heard from his teacher many times what a bisector of an angle is. The words: "bisector of an angle", "to bisect an angle", "two equal angles" etc. were used by the teacher and by the students, were written on the board and even in (part of) the notebooks. Therefore the student knows (even if not aware of it...) that these words describe the issue of "a bisector of an angle" and therefore are worth being used as an answer to a question concerning this issue. If that is the case, they can be thought of as *pseudo- conceptual* answers (*Vinner, 97)*, in which words are connected to words, not to ideas. Moreover, when the student heard these describing words, he had probably built a *concept image* of bisecting an angle. We do not actually know what this image includes and what procedures are connected with bisecting an angle (how he draws the segment, what he thinks of while drawing the "bisector", e.g. bisecting the opposite segment). But it could be even more troubling: maybe there is no procedure at all. Considering the Van Hiele levels (*Hoffer, 83)*, such a procedure occurs in the 3rd level! The student may have a global-visual concept image (based on examples or non-examples of a bisector of an angle). In this case, drawing the bisector does not involve an analytical process but raises the prototype in his head. This time we consider Van Hiele's 1st level - *Visualization*! The teacher, 3rd level interpretations oriented, will probably interpret the students words as having the same meaning he gives them. But do they have the same meaning when used by the student?

5. Alon suggests to overcome the difficulty by confronting the solutions of two problems: If they were drawing median as a bisector they will see now a "real" bisector and find out that they have actually answered a different problem. The teacher uses logical-analytical arguments. He explains that the segment bisecting the opposite side of the angle creates a median and is not necessarily a bisector of an angle. The interviewer reminds him that the student may think in a different way, but the teacher sticks to his logical arguments.

During a simulation of dialogue with a student, he suggests a logical investigation of a conflict. (Interviewer: *They say: "we bisect the opposite side, and that will bisect the angle"*. Alon: *Then let's copy the angle on a piece of paper. If we check, we find out that it's not equal to the other angle. So it's not a bisector of an angle!*). This suggestion, based upon a cognitive conflict, implicitly presumes that a *single* solution to *different* problems would cause: a. rejecting one of the solutions. b. the one to be rejected would be the false one. These two assumptions can be opposed! Can't the student "live with" a same construction (bisecting a segment) which leads to two different concepts (bisector of an angle and median)? Does he really consider them as two different concepts? Or maybe they are two terms, representing the same concept?

6. When Udi is being asked about possible difficulties, he identifies a difficulty of mixing up median and bisector of an angle. He believes that this difficulty is caused by their "similarity". (Udi: *No, the difficulty is not mixing-up with median, but that the bisector of an angle really looks like a median*). This is a "1st level" phrase, and offers global-visual identification of the median which "looks like" the bisector of an angle.

7. Udi believes that the problem exists only in prototypes of triangles, in which the two segments are not easy to be distinguished (Triangles with no extreme attributes: without very sharp angles, without big differences in lengths of sides, etc.). The way the teacher deals with the case of non-prototype triangle (he draws a non-isosceles obtuse-angled triangle and draws two different segments) is probably based on his analytical knowledge, and on the way he and his own teachers dealt with medians and bisectors of an angle (dealing, amongst others, with triangles where two different segments are

needed). Distinction between different cases makes the impression of 3rd level - "from attribute to shape": indeed, different requirements lead to different segments! But, later on, when the teacher draws a bisector of an angle by himself, he abandons this knowledge and returns to first level!

8. Udi reconstructs the drawing from the lesson and draws a "bisector of an angle". Unintentionally, he draws the median. The interviewer asks for his arguments and the teacher gives only verbal arguments of equal angles, incoherent with the drawing (the teacher claims equal angles).

9. The interviewer tries to show him that he was wrong: making an isosceles triangle out of the original triangle (by hiding part of the triangle) should make it easier to see that the segment was not bisector of an angle. But the teacher refers the results to his impreciseness in his drawing. Suspecting that the interviewer meant he was drawing the median he denies. Later on, he gives the reason of "a bad eye" and at last, frankly: "*It will always be difficult for me, it is difficult for me*".

10. The interviewer asks him to draw the bisector of the angle again, but this time "only when he is sure about it". After thinking and concentrating on the angle itself he draws the bisector of the angle. If so, it seems that when there is no intervention, the cognitive processes of the teacher are similar to those of the students. Such a drawing of the bisector corresponds to the first level of Van Hiele, which is based on the prototype of a bisector of an angle in triangles, in which the median and the bisector of the angle are almost the same. Therefore, the learner internalizes the image of a segment which meets the midpoint of the side opposite to the angle. When the teacher is being asked, he knows to give the right explanations ("*you need to get 2 equal angles*") but does not do it in practice. When the interviewer does not give up - he carefully watches his step, leaves the prototype and performs "from attribute to shape".

We notice skipping back and forth between the levels: The teacher explains in 3rd level, performs in 1st level and when asked about his performances - he answers in 2nd level. (He "checks" the attribute of equal angles as was in the prototype, irrespective to its correctness. e.g. when drawing the "bisector of an angle" which does not really bisect the angle...). When the teacher confronts the conflict between ideal and reality he finds it difficult to accept and insists upon: "*I didn't draw a median*". Didn't he?

Last part of the discussion: insight of the teacher and didactic conclusions

1. Udi adopts the process that he was going through as the one to go through with his students. (Udi: *I'll do what you did to me*). We can consider what he says as a recognition of the difficulties he himself had, and the conviction that this way can help solving them. He specifies: Clear non-isosceles triangles should be considered.

2. All of a sudden we feel a double *insight*: a. Udi understands what was wrong. (Udi: *I can tell you what was wrong from the start.. I was following my eye..*). b. He understands that he possibly did not recognize his students' mistakes. (Udi: *They may have done as you said and maybe I saw that and considered it as a correct answer*).

3. While internalizing the difficulty, the teacher is now able to suggest didactic altenatives: Drawing the segments would not help him decide if the students do understand. Therefore, he suggests to observe and listen to them in course of work.

4. Now, that he has become aware of things, he is seeking for the source of the difficulty!

5. Udi likes the interviewers' explanation and is motivated to search for an alternative way, in order to by-pass the difficulty. His reaction while hearing the explanation, may suggest a satisfaction in the possibility of following his students' cognitive processes.

To summarize, we have met Alon, "*a naive idealist*" teacher, with *relational* motivation, who wants his students to understand what they do. He acts at the 3rd level and treats his students as if they were acting at the sáme level.

Udi and Alon, take the students' answers as they are, and believes that they reflect their deeds and their way of thinking. But we face a different situation, since Udi encounters difficulties similar to those of his students, while Alon thinks his students think and understand the same way he does. Naturally, it is difficult for Udi to face his difficulties. Perhaps, the transfer from one level to another, which sometimes enables him to see things from 3rd level point of view, makes it more difficult for him to notice his 1st level way of thinking.

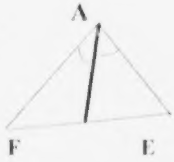
We may point out the beginning of an impressive process of transition to a *constructivistic* way of thinking! Udi is interested in the cognitive processes of his students, and wishes to fit his instruction to the new situation.

c. Extra Lesson About The Bisector of an Angle in a Triangle

The discussion led to an extra lesson about the same subject. The students got a working sheets with two big triangles drawn on it (non-isosceles obtuse triangles). They were asked to draw a median in the first triangle, and a bisector of an angle in the second triangle. They were asked to describe their work. The teachers were walking around, asking some of the students to describe their work aloud. Their purpose was to check what the students understand considering each of the special segments in the triangle (or could we say: what is the students' *concept image* of the special segments).

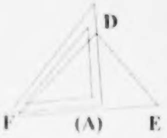
A dialog between Udi (T) and a student (S):

The student, who was asked about the bisector of the angle, answered a reasonable verbal answer (and also pointed at the angle) (S: *I bisected the angle.* T: *How?* S: *I drew a line in the midle of A (pointing)*).



Udi, who became suspicious after the last conversation, decides to check things thoroughly, using the median as a starting point. (T: *And what came out?* S: *Half an angle and another half.* T: *And if you'll be asked to draw a median in the other triangle, how will you do it?* S: *Somehow.*). The student knows the definition of the median and it may seem that he understands! (T: *How do we draw a median from D? What is a median?* S: *It bisects the segment.* T: *Which segment?* S: *FE*).

But his answer considering the median is not enough for Udi. For a reason! Was it only a declarative answer? The teacher uncovers the concept image of the student: (T: *Good. And how do we draw a line in order to be sure?* S: *(Places a ruler and draws a line) T: And after you drew the line? What is equal to what?* S: *F and D are equal to E and D. They are not equal, you have to make them equal*).
(The student used an improper name for a segment (F and D instead of FD).



In his concept image there are equal "things", but they could be different "things":

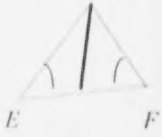
1. It seems that the student seeks for *equality between the two* neighboring *sides* of the relevant vertex. Moreover, the student does not consider the given triangle to be something static, given, fixed, but a *dynamic, variable* thing which can be changed.
2. The median concept image contains division into *two "equal" triangles* (T: *Now you drew the median.... to what have you divided the segment EF?* S: *Into two.* T: *Into two equal segments. Which are the equal segments?* S: *F and D, and E and D (pointing).* T: *Why?* S: *I have bisected them in the middle.... no, let's say that we add some A here (he adds "A"). Then ADF should be equal to ADE (pointing at the two triangles).* T: *Is that what I was asking for? that the two triangles will be congruent?...What would be equal to what?* S: *That FD will be equal to DE(!)*).

Another dialog was between Alon (T) and a student (S1)

S1 built a median and a bisector of an angle in the same way - by bisecting the side opposite the angle. He uses the same procedure to get two different results: bisector of an angle and median. If so, what guides the student to do what he does?

It might be that his only interest is in a necessary adding of a segment to the figure. Necessary, because in the prototype of a bisector of an angle appears additional "line" inside the triangle. Probably, the student was never asked to focus on the angle itself, therefore did not pay attention to different possible results of drawing a segment (e.g., the angle may be divided into equal or non-equal parts). While looking at examples in which there was a "line" inside the triangle, he might have paid attention to other attributes, and noticed, for example, the equality of the angles near the "bisected" angle (equality which was a *non-critical attribute* in the examples he saw!).
(T: *What did you mean to do?* S1: *To bisect the angle.* T: *And what did you get?* S1: *That the two angles are equal.* T: *Which angles?* S1: *This and that (pointing at the angles E and F).*

We may say that drawing a segment is a necessary "line" adding to the "figure", in the context of *global-visual perception* of a triangle with a bisector of an angle (1st level). In this case, the student does not associates the drawing procedure to some "concept attribute" (*does not act in 2nd level*).(S1: *Then I'll write down: "I drew a line from vertex D to the opposite side. Then I formed a bisector of an angle"*) - first, the student draws the line, and only then the bisector is formed!



We may give another explanation to the students' answer about the equal angles E and F. It may be *a pseudo - analytical* answer (*Vinner,97*). A pseudo - analytic solution is using spontaneously a procedure which solved a similar problem, without operating a control mechanism: The teacher and his students have mentioned many times " two equal angles" in the context of a bisector of an angle. The student, who was asked about equal angles finds two angles, the most salient in the figure, and point at them. (Probably, they were already "used" as equal angles in the past).

Moreover, the drawing of the bisector of an angle, on its own, may reflect a pseudo-analytical behavior: vague memories of procedures seen in the past, in which the student "identified" drawing a segment in a triangle, causes him to act the way he did: simply drawing a segment which will create a figure that suits the prototype of "triangle with a segment".

But the teacher uses logical arguments. He tries to convince that one procedure can not cause two different results (3rd-4th level), but that is absolutely not the way that fits the students' level! As we claimed, he does not consider procedures.

Reviewing the teachers' solutions to difficulties arose during the students' work, show a wide set of reactions, based on attention, listening and an attempt to "get into the students' head" and also 2nd-3rd level explanations, which might be hopeless.

d. Discussion with the teachers

We shall point out the main elements that arose in the discussion after the extra lesson:

1. Udi noticed that up to now they were hinting the answer to the students by starting it or indicating the direction. All that remained to do was to complete it. We may say, that was encouraging a *pseudo conceptual* and *pseudo analytical* answers!
2. Udi and Alon noticed that up to this point the students did not have to explain but just to perform. Now when they were asked to explain, they found it almost impossible! We may add that in such "learning" the students did not need to pay attention to the attributes. This manner encouraged 1st level functioning without progress to 2nd level. Moreover, Alon found that the students were *not interested in the process* but only in the final result. We can assume that a *global-visual* referring, a typical 1st level, (which occurs when interested only in the result) causes troubles in analytical description, perhaps because it does not even exist!
3. More specifically, Alon described, enthusiastically (*" for the first time, I noticed that.."*), a student who was looking for some imaginary point on the opposite side of the angle from which to start the bisector of the angle (instead of focusing in the angle "area"). He began to look for explanations of the students' cognitive behavior.(*"In their head, like in the case of height and median, there is some segment emerging from the vertex and has to reach somewhere. They don't understand that in this case, it doesn't matter where it reaches, we don't look at the opposite side!"*).

Conclusion

In this paper we learned about students' concepts and difficulties on one hand, and about the process of learning of the teacher (subject matter, conceptual, pedagogic knowledge), on the other hand. The teachers we were observing, were not attentive to the special obstacles, therefore they accepted the students' answers (which were "logically argued", "fair looking") as correct. Wouldn't many of us react as they did?

Generally, it is hard to identify our own mistakes. But it worth trying! We were following the magnificent dynamics of the teacher being aware of his own cognitive processes, and using them as a didactic lever: he checked things in a different way, suggested to investigate what the students really think, was open to find out the truth about their concept image, and finally, he was eager to know what are the difficulties' sources, looking for solutions. Still, the teachers were not aware of the students' levels of thinking, therefore some of the instructional decisions were faulty.

Could this pedagogical situation be reconstructed and duplicated with other teachers? What will cause the teachers to be open minded, and interested in the cognitive processes of their students and try to adjust flexible didactic solutions? These questions emerge and become guidelines for further research.

Furthermore, being ready to open "Pandora's box", and find out what the students really think, invited "surprises": Which are the equal angles considering the bisector of an angle? And who are the "equals" in median? - segments? triangles? The teachers probably lost their "innocence": It seems that the students' answers were not accepted as they are any more! This is a big step forward. No more "Pseudo- knowledge"! But there was something else: the teachers were ready to go on an independent journey, "searching for the truth" in points that they were not prepared for or thought about in advance (e.g. "what is equal"). The challenge is enormous! If we do not ignore such cases, we should introduce didactic alternatives, which will give answer to the different difficulties of different students. First, we should try to foresee and understand the possible cognitive processes in order to be prepared for them. This is the goal of our broader research.

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Levels of generalization in linear patterns

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In this paper we set forth a proposal about levels of generalization based on students' spontaneous performance during the process of solving linear generalizations problems. The students' acquisition of each level is related to the actual generalization achieved and some features of the students' generalization process are provided. We conclude with a first approach to a genetic decomposition schema of the linear pattern's conceptual structure that students develop when solving linear generalizing problems. We also outline some didactic remarks that should be considered during the teaching and learning process.

Theoretical Background

The study of patterns' generalization in school mathematics has been the focus of research conducted over the last years. Many researchers have made some attempts to investigate stages or levels in the development of patterning ability mainly focused on students' ability to generalize. Stacey (1989) has identified some methods of solution that students use when solving linear generalizing problems. In Orton & Orton (1994, 1996) the adults' and children's answers to questions involving quadratic and linear patterns are classified in stages running from answering questions about concrete numbers to algebraic generalization. Redden (1994) has used the SOLO taxonomy to state two hierarchies of growth concerning first the students' use of data from the questions (Data processing dimension); and second the sense of an overview of the data that can be provided in the form of an expression of generality in the students' pattern description (Expressing Generality Dimension). However, we think that these attempts are mainly focused on the students' written responses to an item and to specific questions within an isolated item. Thus, the dynamic development of learning is not reflected enough. We have also missed a general framework to cope with the problem of students' pattern generalization that can be used not only to classify the students' responses but also to highlight some didactic guides to be used during the process of learning and teaching. Krutetskii (1976, p236) has pointed out that the ability to generalize mathematical material can be considered from two levels: *"(1) as a person's ability to see something general and known to him in what is particular and concrete "* (subsuming a particular case under a known general concept) and *"(2) the ability to see something general and still unknown to him in what is isolated and particular"* (to deduce the general from particular cases). The way the two levels are formulated shows that they do not constitute a hierarchy of students' educational development but both should be seen as educational goals. It is the second level, generalization through empirical induction, the ability we want to develop in students when they are dealing with new problematic situations.

The main goal of our research, we report in this paper, is to state some hierarchical levels of generalization that can reflect the students' performance when dealing with that kind of problems, and can also be used to provide some didactic remarks in helping students to move from one level to the next.

The role played by Reflexive Abstraction (Piaget, 1975) in the generalization's process has been the key feature of some recent research, for instance the action-process-object framework of Dubinsky(1991), and the operative generalization of Dörfler (1991). In our research we have taken the action-process-object framework from Dubinsky, in which the generalizations are constructed through the internal coordination of processes. These processes have their genetic sources in actions performed by the subject on a given stimulus, but we have added the key feature of Dörfler's theory, i.e., a generalization is achieved through the establishment of an invariant which genetic source is again an action performed by the subject. Briefly, a physical or mental action performed by the subject could lead to an internal process, and through coordination or reconstruction (assimilation-accommodation) of existing conceptual schemata, the subject could establish an invariant for the action. The generalization developed could take different forms depending on the actual kind of assimilation of the stimulus by the subject, and therefore different levels related to

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mathematical concept's achievement could be defined. On the other hand, what is actually achieved in any level could be used to derive some didactic remarks to be implemented during the process of teaching and learning.

Methodology

We have conducted our empirical research on a population of secondary education students (15-16 year olds). The first phase consisted in video-recorded interviews administered to eleven students. The second phase was an interactionist teaching experiment with a group of 18 students. Thus, from the interactionist perspective (Bauersfeld, 1994) there are neither pre-given criteria about what is a correct solution nor what constitutes a different solution to a give problematic situation. So students have to contribute to the whole-class discussion providing their own solutions to a problem, and give different solutions from the same problem. They were also encouraged to judge any solution presented to the whole class discussion. Our goal was that these sociomathematical norms (Yackel & Cobb, 1996) could help students to develop a better and deeper understanding of the linear pattern. We think that when a student assumes and uses explanation, judgement or argument as an object itself of discourse, he will require the development of methacognitive abilities that will improve the student's learning outcome.

During the four classroom sessions the students were presented with three linear generalizing problems (stimulus items). These problems underlie a linear pattern, $f(n)=an+b$, being $f(n)>0$, $a>0$, $b\neq 0$, whole numbers. The text format is a word problem illustrated by a drawing of an object and the first three terms of the sequence (number and drawing) are given, i.e., $f(1)$, $f(2)$ *and* $f(3)$, and students were asked to find $f(4)$, $f(5)$ (introductory questions) and $f(10)$, $f(20)$ and $f(n)$ later. Our role was to facilitate and encourage students' participation in small-group and whole-class discussion. When a solution was explained, we had to ask for any other students who wanted to judge it, being careful not to show disagreement or any kind of behaviour that could give any hint to the whole group about the correct or incorrect quality of that solution. Students were obliged to try to develop personally meaningful solutions that they could explain and justify, and reflection upon their own and others' strategies of solutions was encouraged.

Results and Discussion

Some early research (García-Cruz & Martinón, 1997a, 1997b) has provided us with useful information about the students' process of generalization. The actions developed and invariant schemata established during the process of solving a sequence of linear generalization problems are the key feature to achieve a generalization. Also, the conceptual schemata coordinated by the students are important to characterize each level. At each level, we have stated what previous schemata are coordinated and which generalization is achieved by students. Also these levels characterize the cognitive students' behaviour and can be used to distinguish between procedural activity, procedural understanding and conceptual understanding (Zazkis & Campbell, 1996). Our findings are summarized in a final developmental schema that can be seen as a genetic decomposition (Dubinsky & Lewin, 1986) of the linear pattern's cognitive structure through linear generalizing problems.

Level-1 (Procedural activity)

At this level, the student recognizes the iterative and recursive character of the linear pattern, and these are used to calculate the introductory questions. These strategies are not generalizable but are important in highlighting the constant difference of the linear pattern. Such a routine behaviour is later used (another level) when checking the validity of the rules developed. Here students are focused in the most perceptual feature of the pattern: *adding the constant difference* and this action is the only generalization achieved at this level. There is a subtle difference between the "counting all" strategy ($f(10) = f(1)+d+...+d$) and the "counting on" strategy ($f(10)=f(9)+d$): one thing is to add repeatedly the constant difference to get any term, extending the numerical sequence (iterative character) and another thing is to use the recursive character of the pattern using a known term and from this numerical value perform some calculations to get the required term. The term procedural activity could be used to characterize the student 's behaviour at this level.

Level-2 (Procedural understanding, Local Generalization)

At this level, the student has established a local generalization. This means that he or she has been able to establish an invariant from an action performed on the picture or numerical sequence, within any new problem given, although this invariant could be different from problem to problem. The establishing of the invariant means that the same calculation rule, derived from actions to calculate a specific term, has been applied to any other calculation within the same problem or situation (García-Cruz & Martinón, 1997a).

The establishing of an invariant also means that the stimulus has been assimilated and accommodated within an already existing cognitive schema, i.e. indirect counting methods, function (as a process), and proportional reasoning. The existing cognitive schema is identified by the student 's written of verbal response.

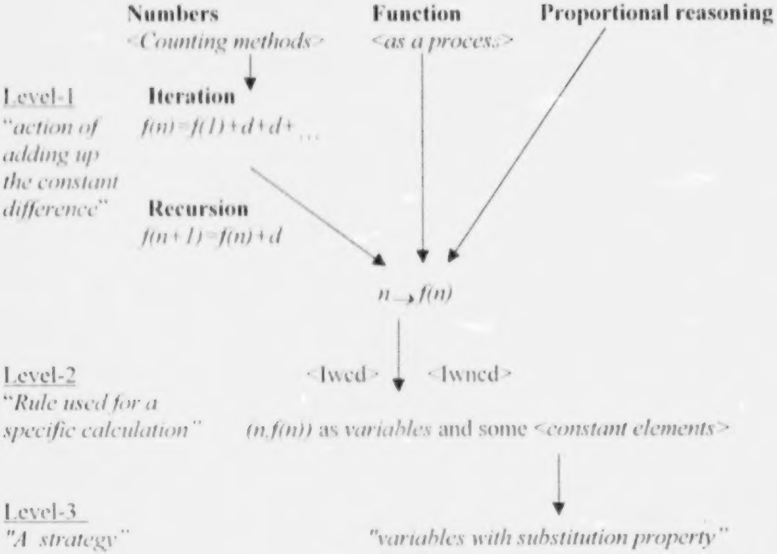
The student can also establish an incorrect invariant because the stimulus is assimilated to an incorrect cognitive schema, i.e., proportional reasoning. As we said above the establishing of an invariant is detected through the calculation rule used by the student in any question within a problem. If the canonical form of the linear pattern is f(n)=5n-1, then the assimilation of that stimulus to the incorrect cognitive schema of proportional reasoning could lead the student to the establishing of an invariant of the form $f(2n) = 2f(n)$. Later through checking and adjustment, this invariant could take the form $f(2n)=2f(n)-1$ which is valid only for even terms in the sequence. The student's attention can be also focused on some relations and connections between some elements of the drawing, and as a result an invariant of the form $f(n) = 6n-(n-1)$, or $f(n) = 6+5(n-1)$ can be established. So in establishing an invariant, students confer a variable quality to $f(n)$ and n, i.e., value of the term and position occupied in the corresponding sequence be numerical or pictorial.

The key feature here is that a shift from procedural activity to procedural understanding has taken place, and this shift can be clearly observed in the students performance. Thus, what has been generalized here is the *specific rule* for a calculation. This rule has always variable and non-variable elements, and the character confered to the variable elements should be taken as a generalization. Indeed, an extensional and intensional generalization has taken place, because the specific elements, numerical or pictorial, used to develop the rule have been detached from their initial meaning and their reference range has been extended. When the established invariant is correct the term procedural understanding could be used for the student's cognitive behaviour.

Level-3 (Conceptual understanding, Global Generalization)

At this level, the student has generalized a *strategy*. That means that he or she has performed the same action and established the same invariant in a new but similar problem. The *rule* developed and used in an early problem is now an object which serves as an stimulus for an action: apply or transfer the action performed and invariant established in another problem to a new problem which has been recognized as similar to other already known. At this level, what is achieved as a generalization is the student's overall performance when dealing with these situations, and this is what we call a *strategy*. So a strategy has the action and the invariant established as components in a particular situation. Now this strategy is used in a new but similar situation. The constant elements, if any, (which are present in the syntactic structure of the invariant) acquire the quality of variables through that process because they loose their constant character and are substituted by different numbers. The students cognitive behaviour could now be considered as conceptual understanding.

The following schema summarizes the above discussion and could be considered as a first approach to the genetic decomposition of the students' conceptual structure of linear patterns developed spontaneously by students though linear generalizing problems (Dubinsky & Lewin, 1986).


Numbers
<Counting methods>
Function
<as a process>
Proportional reasoning
Level-1
"action of adding up the constant difference"
Iteration
f(n)=f(1)+d+d+...
Recursion
f(n+1)=f(n)+d
n → f(n)
Level-2
"Rule used for a specific calculation"
<Iwcd>
<Iwncd>
(n,f(n)) as variables and some <constant elements>
Level-3
"A strategy"
"variables with substitution property"

(Note: *Iwcd* and *Iwncd* stand for Invariant with constant difference and Invariant with no constant difference respectively)

The above schema should be taken as the ways in which students spontaneously develop their understanding of linear pattern's conceptual structure, but perhaps we think it is not complete.

From the teaching experiment we have drawn some conclusions.

First, it takes time before a student realizes that the existing conceptual structures are not sufficient to assimilate the new problematic situation, so many students keep on establishing incorrect invariant and it seems for us very difficoult to remove this unsuccessful behaviour. For those students we strongly recommende the generalization of conditions for actions (second process of Dörfler's theory), so in order to carry out actions these are better if the constant difference are a key component.

Second, other students are successful in establishing an invariant (local generalization) but they move from one invariant to another (even incorrect) when confronted with a new situation. For those students we strongly recommend the generalization of the results of actions (third process of Dörfler's theory).

Third, once a local or global generalization has been achieved, the students should be confronted with a large number of new situations before the new cognitive structure becomes stable and permanent.

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THE EVOLUTION OF PUPILS' IDEAS OF CONSTRUCTION AND PROOF USING HAND -HELD DYNAMIC GEOMETRY TECHNOLOGY

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Abstract

This paper considers how the use of hand-held dynamic geometry software can contribute to the development of pupils' understanding of ideas associated with construction and proof. In adopting a socio-cultural perspective, the technology is seen as a mediating tool and intellectual development as a complex, dialectical process. Classroom research is reported on involving a group of Year 8 pupils (aged 12-13) in a mixed urban comprehensive school in the North of England during the autumn term of 1997. The data analysis is undertaken with particular reference the Vygotskian notions of 'spontaneous' and 'scientific' concepts. It is suggested that such a perspective helps to illuminate the potential of the technology in supporting the complex and dialectical process of developing ideas of construction and proof.

Introduction

The classroom research reported on in this paper is part of a wider study with the aim of investigating the potential of hand-held dynamic geometry software in the secondary school classroom. The research and development has taken place using Cabri on the Texas TI 92 calculator. The focus of the paper is on how the use of such technology can contribute to the development of pupils' understanding of ideas of construction and proof.

Background literature

Dynamic geometry software, seen as a mediating artifact, provides an environment, which supports mathematics learning as highlighted by Jones (1996). The TI 92 has the particular characteristic of enabling the development of a desktop environment in which the dynamic geometry environment (DGE) can be one mediating artifact used alongside more traditional tools.

Healy et al (1994a) illuminate the way in which dynamic geometry can be introduced using the drag function to emphasise the difference between drawing and construction, and proceed to consider constructions in particular further (1994b). Hoyles et al (1995) consider the interdependence of construction and proof and the replacing of proof by construction in a dynamic geometry environment.

In developing the use of the DGE in this study, the drag function and the idea of a construction invariant under drag have been central. The associated classroom materials, which have been developed are directed at making a distinction between drawing and construction, and at seeking an understanding of concepts such as that of using a circle to preserve length (Healy et al, 1994a, 1994b). Pupil fluency with the technology has been a further central consideration as highlighted by Goldstein et al (1996).

The work of Fischbein (1982) is considered to be relevant to this study. He identifies three forms of conviction; formal, arising from argument, empirical arising from a number of practical findings, and an intuitive intrinsic conviction, which he calls 'cognitive belief'. It is suggested that the DGE can reflect these ideas through dragging to test constructions, dragging to provide empirical proof and also through children's intuitive ideas which are triggered by the use of the DGE.

Theoretical Framework

This study is framed within a Vygotskian perspective. Such a perspective places emphasis on the idea of mediation by a variety of tools, as highlighted by Jones (1996) in a similar environment, within the zone of proximal development (Vygotsky, 1962). Vygotsky originally defined the zpd in terms of development whilst more recent definitions, found to be relevant to this study, have related the zpd to activity theory (Engeström, 1987) and to the teacher and class as a whole (Hedegaard, 1990).

The interplay between *everyday* and *scientific* is also considered relevant:

> 'In working its slow way upward, an *everyday* concept clears a path for the scientific concept in its downward development. It creates a series of structures necessary for the evolution of a concept's more primitive, elementary aspects, which give it body and vitality. *Scientific* concepts in turn supply structures for the upward development of the child's spontaneous concepts toward consciousness and deliberate use.' (Vygotsky, 1962)

In considering the notion of development, Confrey (1995) highlights this as follows:

> 'Development conceived of as a complex, dialectical process characterised by a multifaceted, periodic timetable … by a complex mixing of external and internal factors, and by a process of adaptation and surmounting of difficulties.'

Confrey argues the need for an historical analysis and that one must examine the growth of higher mental functions in order to understand them.

Methodology

From a Vygotskian perspective it could be said that methodology should not only be all-pervasive in a study, it should be the study.

"The attempt to categorize Vygotsky, to 'dualize' him as either a psychologist or a methodologist, contradicts, ironically, not only Vygosky's life-as-lived, but his self-conscious intellectual revolt against dualism" (Newman and Holzman, 1993, p 16). Vygotsky can be seen as a methodologist/ psychologist in the sense that his all-embracing view of the science of learning brings in the Marxist historico-cultural dialectic and the ideas of revolutionary activity and practice. It provides a methodology, which informs and pervades a study and is available to constantly influence the conclusions drawn and the direction of future progress.

This methodology is echoed in the idea of "tool-and-result" outlined by Newman and Holzman (1993 p38), who draw a distinction between tools such as hammers and screwdrivers (tool for result), and dies and jigs (tool-and-result). Hammers and screwdrivers are bought and used as needed, dies and jigs are tools designed and refined by the worker. Vygotskian methodology is a 'tool-and-result'. Like the jig, it is bound up in its result.

Found to be consistent with such an approach have been ideas drawn from Grounded Theory (Strauss and Corbin, 1990). This involves the systematic process of review and refinement to allow the simultaneous development of theory and collection of data and for a progessive focussing on the emerging issues.

Data Collection

The classroom research so far has taken place in two phases. In the first phase the teacher/researcher taught a class of 30 Year 7 pupils (aged 11-12). The second phase involved working with a group of Year 8 pupils (aged 12-13). Both phases were carried out in mixed urban comprehensive schools in the North of England during 1997. The classroom research has involved the development of materials, which have the aim of releasing the potential of the dynamic geometry software and which, at the same time capitalise on the hand held nature of the T I 92. The development of the materials was guided by the ideas of Hedegaard (1990) and in particular the notion of a 'whole class zpd' in which the role of the teacher in relation to the class as a whole is emphasised. This development has been against a backdrop of the desktop environment where a hand-held DGE has been shared between pairs of pupils in order to stimulate collaboration and interaction.

Data Analysis

This paper reports on the second phase of the classroom trials. The pupils had not used the T I 92 before and met with the teacher/researcher in their lunch breaks. After a brief period of familiarisation, they were given a task of constructing a square, which was stable under drag. The following fragments from the resulting dialogue are presented below. The three pupils involved are Ryan, David and Joanne and the teacher/researcher is JG.

The pupils had been allowed to take the machines home and Ryan had seen the 'Regular Polygon' option, which allows 'construction' of a square directly. This extract is from the following session.

1. D Does anyone know how to draw a square?

2. R Polygon, Regular Polygon

The 'Regular Polygon' option offers a hexagon first and it is not immediately evident how to draw a regular polygon with fewer sides. In this case the use of the technology was not that helpful in assisting pupils to develop their ideas about construction.

Joanne also had taken a machine home and her explorations had led in another direction, towards the 'measuring' menu.

3. J I'm doing it on normal polygon it's a lot easier and you can always measure
4. your lines.

5. D It's hard to get it a proper square.

6. J But afterwards you can measure your lines.

7. R Yeah you can can't you

8. D I know! You could do it two triangles, two right angled triangles next to each
9. other and merge them, then it'd be a proper square.

10. R I think I've got a perfect square here.

11. J See, I've just figured out mine's not right, cos one of my lines is 1.91 cm and
12. the other is 2.03 cm

13. J There's also area; you can do the angle and see if the angle's a right angle, as
14. well.

15. R Well you can tell if it's a right angle.

16. J Yeah but you can't for definite

17. D I think it is regular polygon.

All three went on to use regular polygon successfully and dragged their squares.

Joanne's investigation led her towards attempts at simply drawing the square. However the development of ideas of construction as distinct from drawing become evident from this interaction. For example, David makes reference to a 'proper square' at line 9 and Ryan talks about a 'perfect square' at line 10. These examples could be viewed as evidence of these pupils' spontaneous conceptions of the idea of construction (of a square in this particular case). The discussion at lines 15 and 16 centres on different levels of conviction. For example, Ryan suggests that 'you can tell if it's a right angle' which Joanne counters with the comment that 'but you can't for definite'.

In the second part of the exercise the group were asked to carry out the same task, but not to use polygon or regular polygon. David (Figure 1) and Ryan (Figure 2) both used a circle, a radius and a perpendicular through the centre, as a starting point. Ryan had defined a point where he estimated the other corner of the square to be and drawn two rays through that point. David had drawn two segments, again by eye, to complete his square. Dragging showed him that the point was not defined. Ryan drew two angle bisectors, which coincided originally but separated if he dragged the undefined corner of the square

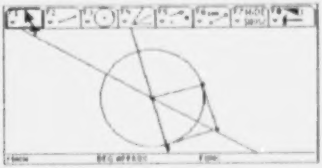

Figure 1 (David)

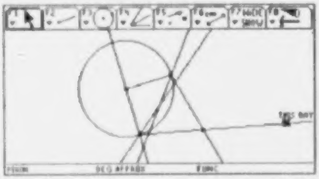

Figure 2 (Ryan)

This conversation followed.

18. D I'm trying to do an angular bisector... cos if the angular bisectors make a right
19. angle in the middle then that'll mean it's a square, but I can't get it to do them.

20. JG How do you know that the angle bisectors will meet in the middle in a
21. right angle?

22. D Well I don't know that they will in a right angle

23. R They will

24. D If it's a proper square then it'll be in a right angle because you'd be chopping
25. the square like diagonally

26. R There'd be four triangles

27. D There'd be like four triangles and they'd all be right-angled triangles

28. R There'd be two 45° angles

(Shown how to draw angle bisector)

29. D Yes!! Now that looks like its going at a 45° angle right through.

30. That meets in the other corner there, so I think that means it's a square.

Once again, it is suggested, there is an interplay between different levels of conviction and mathematical argument. In the passage from lines 22-28 a sufficient definition of a square is arrived at eventually, only to be abandoned at line 30 for the germ of a new approach.

Joanne used a different starting point. She began by drawing a line segment and was wondering how to continue.

31. JG So you've got one line like that.... What would help you to draw a
32. square?

33. J It would have to be parallel

34. JG So you want to draw a line parallel to this...

35. J Yes

36. JG and where does it have to be?

37. J It has to be the same length as that down

38. JG Like that? So how would you draw it? What shape would help you draw
39. that down to there?

40. J A triangle

41. JG Look on F3

Joanne chose the circle option and went on to successfully draw a square (Figure 3)

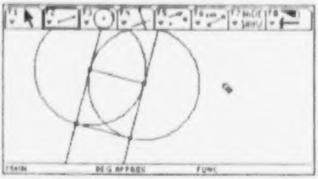

Figure 3 (Joanne)

There followed an attempt to probe understanding of geometrical isometries. This conversation refers back to Figure 2.

42. R I dunno. If I try dragging this ray, because the ray's not secure at the point,
43. that ray'd drag around wouldn't it?

44. But if that was a perpendicular to that ray,……

45. JG So this circle is a good starting point isn't it

46. If you have that circle and that ray, how many sizes of square can you draw?

47. R just one

48. JG as soon as you've drawn that and that

49. R Once you've drawn the circle then you've got the size

Ryan went on to construct a square by drawing a circle (Figure 4), a ray from the centre and a perpendicular through the centre, followed by two perpendiculars where the first two lines intersected the circle.

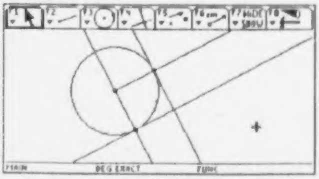

Figure 4 (Ryan)

Pressing the grab key when the cursor is away from the diagram makes the independent points in the diagram flash. This useful facility allows pupils to explore geometrical isometries.

50. JG What flashes?

51. R That corner there. (The centre of the original circle)

52. Does that mean that's the only corner that can be dragged?

53. JG That's the only point that can be dragged. Tell me what you drew first.

54. R I drew the circle first

Ryan went on to discover that he could grab the circumference of the circle as well as the centre and so alter the size of the square, and alter the orientation of the diagram by dragging the original ray. By a similar process Joanne realised that the original line segment in her diagram completely defined her square.

Discussion

In observing this classroom activity and in analysing this interaction, there is a clear interplay between ideas of drawing and construction and also between notions of necessary and sufficient conditions (for construction). It is argued that this interplay reflects that between pupils' spontaneous concepts and their developing ideas related to scientific concepts, which in this case are associated with ideas of construction and proof. These pupils can be seen to be operating in a dialectic between their spontaneous conceptions of proof, informed by their ideas and the insights available to them via the mediating role of the DGE and other desk top tools, and the scientific concepts of construction and proof. The second episode in particular provides a rich illustration of how *everyday* (spontaneous) concepts 'create a series of structures necessary for the evolution of a concept's more primitive, elementary aspects, which give it body and vitality' and hence how *scientific* concepts 'in turn supply structures for the upward development of the child's spontaneous concepts toward consciousness and deliberate use' (Vygotsky, 1962). By the end of this episode, it is suggested that both Ryan and Joanne have displayed evidence of an appreciation of the idea of construction and that they have had at least an elementary introduction to ideas associated with geometrical isometries.

It is further suggested that this development can be seen to parallel Fischbein's (1982) three forms of conviction; formal, arising from argument, empirical arising from a number of practical findings, and an intuitive intrinsic conviction or 'cognitive belief'.

It is also suggested that this interaction is illustrative of development 'conceived of as a complex, dialectical process characterised by a multifaceted, periodic timetable ... by a complex mixing of external and internal factors, and by a process of adaptation and surmounting of difficulties' (Confrey, 1995).

In considering this process of development, the role of the teacher within the zpd has been found to be an essential element in assisting pupils to move from their spontaneous/everyday conceptions towards more scientific concepts. This echoes the findings of Jones (1996) who argues the need for a significant input from the teacher when pupils are working within a DGE.

A further aspect of the wider study, for which there is little room in this paper, to consider in any great depth has been the interplay in the desk-top environment between the DGE and the traditional tools such as pencil and paper.

A final observation in relation to the aims of the wider study is of the undoubted potential of such hand-held dynamic geometry environments to promote the development of pupils' understanding of notions of construction and proof.

Thanks are due to pupils and staff from King Edward VII School, Sheffield and Mossley Hollins School, Tameside.

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COGNITIVE UNITY OF THEOREMS AND DIFFICULTY OF PROOF

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The cognitive unity of theorems - a theoretical construct originally elaborated to interpret student behaviour in an open problem solving holistic approach to theorems - was transformed into a tool that may be useful for interpreting and predicting students' difficulties when they are engaged in proving statements of theorems. The aim of this paper is to explain (through "emblematic" examples) the potentialities of this tool and indicate possible further developments concerning both research and educational implications for the approach to proof in schools.

1. Introduction

In preceding papers regarding the approach to geometry theorems in school (Garuti & al.1996; Mariotti & al.1997), a specific theoretical construct ('**cognitive unity** of a theorem') was introduced in order to stress the importance of a holistic approach to theorems and to interpret some of the difficulties met by students in the traditional approach to proof. Cognitive unity of a theorem is based on the **continuity** existing between the production of a conjecture and the possible construction of its proof.

The idea of this construct initially came from the *epistemological* analysis of work done by past and present geometers, which revealed many examples of continuity between the production of a statement and the construction of its proof, in particular as concerns the relationship between, on the one hand, specifying the objects of the conjecture, determining stricter hypotheses or stating a new weaker conjecture and, on the other, performing trials to prove the statement (Lakatos,1976; Thurston, 1994). We then found a *cognitive counterpart* of this analysis: in a teaching experiment concerning the production of theorems (conjectures and proofs) by beginners in a mathematical modelling problem situation concerning sunshadows, we found experimental evidence of the cognitive unity between the phases of conjecture production and proof construction. We expressed this unity in the following terms:

> (CU) "*during the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organising some of the previously produced arguments according to a logical chain.*".

These findings led us to state that in order to bring about a *smooth* approach to theorems in school, it is necessary to consider the connection between conjecturing and proving, in spite of the undeniable differences between these processes. We also wondered whether in a traditional school approach to theorems one of the difficulties that students face could be that of reconstructing that unity, when it happens to be hindered. A very particular, we should say extreme, situation is that of the task *"prove that ..."* : in this case the process of conjecturing is not demanded and the unity is broken. Unity may be reconstructed only by the riappropriation of the statement through a new process of exploration, i. e. reconstructing the whole cycle: exploring, producing a conjecture, coming back to the exploration, reorganizing it into a proof.

These reflections suggested the following questions: *Can the cognitive unity construct be a tool allowing teachers and researchers to predict and interpret students' difficulties when they have to prove a* given *statement? Can it be a tool that allows the teacher to select appropriate tasks that increase in difficulty, in relation to the increasing difficulty in establishing continuity between the statement and the proving process?* The study reported in this paper aims to produce some partial ansvers to these questions.

2. Towards a Tool for Interpreting and Predicting Difficulties in Proving

In the aforementioned teaching experiment, we analysed the behaviour of all the students who had produced *mistaken conjectures*. When they had to prove the statement finally agreed on in classroom discussions, they were in the same situation as the students who have to prove *a statement they themselves had not produced* (that is, in the same situation as the traditional school approach to proof -see 1.). We discovered that all the students who had successfully managed the proving activity appropriated the conjecture to be validated through a dynamic exploration of the problem situation; they produced arguments for the conjecture's plausibility, which then were useful for proving the statement (a process similar to the production and validation of an *original* conjecture, according to the analysis performed by Boero & al, 1995). The activity of dynamic exploration and search for arguments for plausibility appeared to be a necessary step towards the construction of the proof.

Research work by Simon (1996) and Harel (1996) suggested that the students' proving processes should be considered from another wider perspective. Simon describes the possible role of *"transformational reasoning that involves envisioning the transformation of a mathematical situation and the results of that transformation"* in *"theorem generation, making of connections among mathematical ideas and validation of mathematical ideas"*. Harel points out the role of *"transformational proof schemes"* as *"foundation for all theoretical proof schemes"*. Thus, Harel's and Simon's work suggested us to take into account the different kinds of *transformation (of what? in relationship with what?)* that can intervene during the proving process. Indeed, we may consider different objects and different levels of transformation: a purely syntactic, but goal oriented transformation (like in some proving processes based on algebraic transformations - see Boero, 1997); a transformation of the situation represented by the statement, in order to generate another statement, easier to prove; a translation into another language (for instance, from verbal to algebraic language), etc. *"Dynamic explorations"* considered by Boero & al. (1996) are based on suitable imagined, or concretely performed transformations of space configurations. We also note that a proof (the final product of an effective proving process) may be regarded, by itself, as a chain of transformations of the statement according to logical rules. Further possibilities of *"transformational reasoning"* in proving are suggested by Polya's work concerning problem solving, combined with the idea of "mathematical theorem" as a statement, its proof and the reference theory (see Mariotti & al, 1997). Polya pointed out that in some cases it is very profitable to perform the global *"transformation of the problem"* to be solved by setting it in a theory different from that in which the problem was originally conceived. As an example of transformational reasoning in proving applied to the reference theory, let us consider the history of the proof of the Fermat's last theorem. In this case exploration of the statement in the field of arithmetics did not produce arguments which might be immediately exploited to construct the proof; construction of the proof called for a major transformation of the statement (by interpreting it as a particular case of a conjecture concerning a different field of mathematics- a change of reference theory) together with a jump in the complexity of the elaboration of appropriate arguments.

Taking into account these reflexions and our preceding work, finally we reconsidered our theoretical construct of the cognitive unity of a theorem (see CU) in order to get a pointer of the difficulty of proving a *given* statement and, consequently, as a tool to predict the level of difficulties met by students. We defined as *gap between the exploration of the statement and the proving process* the distance between the arguments for the plausibility of the conjecture, produced during the exploration of the statement, and the arguments which can be exploited during the proving process. In some cases the gap remains inside a reference theory; in other cases (see Fermat's theorem), the gap concerns also the transition from arguments in one reference theory to arguments in another reference theory. At this stage of our research, we may formulate the following tentative hypothesis:

> *the greater is the gap between the exploration needed to appropriate the statement and the proving process, the greater is the difficulty of the proving process.*

Taking into account the preceding reflections, we may say that this *gap* may be reduced through suitable *transformations* (concerning the formulation of the statement, and/or the situation represented by it, and/or the reference theory, etc.); the *exploration of the situation described by the statement* and these *transformations* appear as necessary ingredients of the construction of the proof of a *given* statement.

We shall present some examples that illustrate and support the validity of this perspective *in the case of proofs of given statements which do not need changes concerning the reference theory*. These were chosen in a non-geometrical field (the elementary theory of numbers) in order to avoid our perspective, which was elaborated in the geometrical domain, being regarded as context specific.

In spite of all the experimental evidence collected till now, we think that an important work remains to be performed, i.e. the final, precise formulation of our hypothesis, its verification, its integration in the perspective outlined above (where the idea of "*transformation*" in interplay with the idea of "*dynamic exploration*" plays a major role) as well as the implementation of its educational implications. These topics will be outlined in Section 5.

3. Construction of Proofs of Given Statements: Some Examples

The following examples were produced by students ranging from grade VII (lower-secondary school) to undergraduate level. They have been chosen as representative cases, so their comments point out general aspects. All the examples concern students accustomed to dealing with open problem situations and reporting in detail their reasoning in written form. All the tasks are of the type "*Prove that...*"

3.1. "*Prove that the sum of two consecutive odd numbers is a multiple of 4* "

Two examples of proof are reported:

a) a grade X student

« I can write two consecutive odd numbers as $2k+1$ and $2k+3$, so I will find: $(2k+1)+(2k+3) = 2k+1+2k+3 = 4k+4 = 4(k+1)$.
The number I get is a multiple of 4».

This student quickly appropriated the statement; he wrote down the sum of the two consecutive odd numbers in a suitable way (by a translation from verbal language into algebraic language) and then performed suitable standard algebraic transformations; the interpretation of the final formula validated the statement.

b) a grade VII student

« I shall perform some tests: $3+5=8$; $1+3=4$; $5+7=12$; then I can write these additions in this way: $3+5=3+1+5-1=4+4=8$ (the same for the other additions). It is like adding the even number in the middle position to itself, and the double of an even number is always a multiple of four".

In this case the student does not know algebraic language, so he needs to explore the statement in order to transform it. The exploration shows the equivalence between adding two consecutive odd numbers and adding two appropriately chosen equal even numbers. The interpretation of the result of the performed transformation (*"double of an even number.."*) allows the validation of the statement. We may remark that the student moves inside the frame of a very elementary theory of numbers as "reference theory" (appropriately exploiting properties like "*The sum of two even numbers is even*").

In spite of the differences between the two proving processes (usage of algebraic language vs natural language), in both cases there is continuity between the appropriation of the statement and the construction of the proof, and the transformation of the statement is not a difficult task.

3.2. "*Prove that the number $(p-1)(q^2-1)/8$ is an even number, when p and q are odd numbers* " (following an idea by Arzarello, 1993).

The proof chosen as an example was produced by a fourth year university student (in a mathematics education course concerning problem solving). In order to facilitate the analysis, the student's text is subdivided into "episodes":

Ep. 1: *« p and q are odd, then $p=2m+1$ and $q=2n+1$ ».*

Ep. 2: *« I shall analyse the formula: $(p-1)$ is even, q^2 is odd, then (q^2-1) is even, so $(p-1)(q^2-1)$ is even, being a product of even numbers. But in this way I get no result because in general it is not true that an even number divided by another even number makes an even number.*

Ep. 3: *« I shall try a transformation:*
$(2n+1-1)[(2m+1)^2-1]/8 = 2n(4m^2+4m+1-1)/8 = 2n\,4\,(m^2+m)/8$ »

Ep. 4: *« if $p=1$ e $q=3$ then $0*8/8=0$; $p=5$ and $q=7$ then $4*48/8=24$; $p=11$ and $q=13$ then $10*168/8=210$. It seems that by substituting q with an odd number, q^2-1 is always divisible by 8. If I succeed in proving this in general, everything is* *fine, because at this point I would get an integer number multiplied by an even number (that is,* $p-1$*) and so it is obvious that the result is even.»;*

Ep. 5: *« Now I shall prove that, if* q *is odd,* q^2-1 *is always a multiple of 8;* $q=2n+1$ *then* $q^2-1=4n^2+4n+1-1=4n(n+1)$*. This is at least divisible by 4, and so what remains is* $n(n+1)$*, which is surely divisible by two, because if* n *is even everything is fine, if* n *is odd then* $(n+1)$ *is even. We may conclude that* q^2-1 *is a multiple of 8»;*

Ep. 6: *«I know that* $(p-1)(q^2-1)/8$ *is even if* p *and* q *are odd. The conclusion arrives quickly after the illumination that* q^2-1 *is divisible by 8».*
(our underlining).

Analyzing this student's performance, we may remark that:
- the first episodes (1, 2 and 3) apparently lead nowhere, but they serve as non-goal-oriented exploration of the statement. It is as if the student is "testing the ground" to find something, but without knowing what. The meaningful passage appears in Episode 4, and the manner in which it arises is typical of conjecturing: numerical tests, observation of a regularity which leads to a conjecture. The underlined sentence is illuminating: it is rather frequent, during exploration of a statement, to arrive to this point: "*if I could prove B, then I would have proved A*" (crucial lemmas are frequently generated in this way). From this moment on, students' operations are goal-oriented and intended-anticipatory (see Harel, 1996); that is, they aim *«to derive relevant information that deepens one's understanding of the conjecture and potentially leads to its proof or refutation»*
- exploration of the statement leads the student to generate a new theorem ("q^2-1 *is a multiple of 8, if* q *is odd*"). This aspect, which we consider particularly interesting, confirms what Simon (1996) observed, although in different situations, about *trasformational reasoning* (see 2., quotation from Simon).
- between Ep. 4 and Ep. 5 we may observe, from the student's subjective point of view, a change in the status of the sentence: "(q^2-1) *is divisible by 8*". Indeed, at the beginning it is considered as a conjecture; the student is not sure about its truth and writes "*It seems that*"; it then becomes a statement to prove: the student starts Ep. 5 by writing "*I shall prove that*". The same behaviour had been observed also in a very different situation, with 8th-graders (see Garuti & al, 1996).

In general, we may say that this student appropriates the statement by transforming it and establishing continuity between the exploration of the transformed statement and the proving process. Metaphorically we may say that through the exploration of the statement the student tries to unravel a tangle, and then by following the thread builds up the web of proof. In our opinion in the case of this theorem the gap is greater than in the preceding case, because the appropriation of the statement needs a more complex transformation and finding the "thread" (which allows continuity) is more difficult. In order to support this hypothesis, we may consider the behaviour of students who fail to costruct the proof. Some of them meet difficulties in interpreting the same formula reported in Episode 3, which was obtained by them through standard algebraic transformations; some of them (in episodes similar to

Episode 2) think they have proved the statement by writing that the product of $(p-1)$ and (q^2-1) is always an even number

3.3. "*Prove that if two numbers are prime to one another, the sum will also be prime to each of them* " (Euclid's Elements, Book 7, Prop. 28; taken from Heath, 1956).

Also in this case, the chosen example concerns a fourth year mathematics student.

Ep.1 :« *$GCD(a,b)=1$ then $GCD(a,a+b)=1$ and $GCD(b,a+b)=1$*
I shall try to reason by contradiction: if $GCD(a,a+b)=c$ with $c \neq 1$, then $a+b=cn$, consequently $(a+b)/c=n$, that is $a/c+ b/c=n$
I can say nothing because for instance $1/2+1/2=1$, but 1 is not divisible by 2»;

Ep.2 :« *As a is divisible by c and $c=GCD(a,a+b)$, it follows that c divides both a and $a+b$. Then $a/c=m$ and then $b/c=n-m$, that is c divides b: absurd!*»

Ep.3: « *I think that this is true; I shall try to formalize it better:*
$GCD(a,a+b)=1$ with $c \neq 1$; *then $a+b=cn$, $a=cm$;* **[*]**
$(a+b)/c=n$, $a/c+b/c=n$, $m+b/c=n$; *$b/c=n-m=m'$ then $b= cm'$*
a and b have at least c as a common divisor, but then $GCD(a,b)\neq 1$ and this is absurd. The idea of reasoning by contradiction came to mind because I consider it natural when I have to prove things of this kind.».

Analysing this proving process we may note that:

- in Ep. 1 the student translates the statement into symbolic language and transforms "*to be prime to one another*" into $GCD(...)=1$, then performs standard algebraic transformations leading him nowhere; note how his interpretation of "$GCD(a,a+b)=c$ *with* $c \neq 1$" in terms of "$a+b=cn$" is only partial!
- full appropriation of the statement happens in Ep. 2, when the student interprets the property formally expressed by "$c=GCD(a,a+b)$ " as "*c divides* *both a and a+b*", and then translates this statement into formulas which allow an easy and effective algebraic transformation. This very passage will allow him to prove the statement by continuity with the preceding exploration;
- the passage marked with **[*]** is the missing link in Ep. 1; this passage becomes explicit only after the exploration performed in the Ep. 2;
- performing a proof by contradiction presents no difficulty for this student as for other students in the same group.

In general, we may remark that exploration of the statement is made difficult by the fact that the statement encapsulates a non-trivial "logical" content. The gap between the exploration of the statement and the proving process is relevant: exploration may remain at the level of formal transformations of the statement (for instance, from "*to be prime to one another*" to "$GCD(...)=1$" to a contradiction: "*if* $GCD(...)=c$ *with* $c \neq 1$") without fully penetrating the meaning of the statement. On the contrary, in the preceding case 3.2. an effective exploration was possible through standard algebraic transformations or conjectures about numerical cases.

Once again, our interpretative hypothesis is confirmed by students who failed the proof: they made numerical trials (which provide arguments for plausibility), they performed formal transformations (like at the beginning of Ep. 1), but they did not succeed in penetrating the logical knots of the statement; for instance, some of them, while reasoning by contradiction, erroneously supposed that "*a is a multiple of b* "(or "*b is a multiple of a*"), failing the interpretation of "*to have common divisors*".

4. Returning to a Preceding Experiment: A Deeper Interpretation

Our hypothesis concerning the cognitive unity of a theorem allows a deeper interpretation of what happened in a preceding teaching experiment, described in Boero & al. (1995). Seventh grade students had produced (through exploration of numerical examples) two different formulations of the same property:

a) "*A number and the number immediately after have no common divisors except for the number 1*" (We called this a "relational statement").

b) "*If you add 1 to a number, all its divisors change, except 1*". (We called this a "procedural statement")

We observed how in this particular case the different formulations of the statement influenced the proving process: students who referred to the relational statement were not able to go beyond exploration of the conjecture. Indeed they considered, in some numerical cases, the divisors of a number and the divisors of the following number, observing that there was no common divisor, with the exception of 1. No general proof was constructed.

On the contrary, some students referring to the procedural statement were able to construct a proof; they considered the divisors of a given number, then they transformed it into the following number and checked if the divisors of the first number divided also the second, discovering that the added unit constituted the remainder of the division of the increased number by the divisors of the initial number (and so they developed an appropriate, general argument for a proof).

These different behaviours led us to hypothetize the existence of a "textual continuity" between the statement and the proof. We now believe we can interpret those students' behaviours in a deeper and more appropriate way: the gap between the exploration of the statement and the proving process is less with the second formulation (the exploration provides a suitable, crucial argument for the proof).

5. Concluding Remarks and Further Developments

At this stage of the research we can say that, from the educational point of view, the teacher can use the construct of the cognitive unity as a tool for predicting and analysing some difficulties met by students when they have to construct a proof. In particular, the way a statement supplied by the teacher is formulated is of relevance, especially for beginners (see 4.). But it is also important that students gradually learn to transform autonomously the given statement in order to establish a continuity between exploration of the statement and construction of proof (indeed, in an example like that discussed in Section 4. students can obtain an easy proof by transforming the statement). This remark confirms the importance of *transformational reasoning* and the necessity of *nurturing* it (Simon, 1996, p. 207). The problem of how to implement this indication in class work remains still open!

From the research point of view, let us consider the case of proofs needing a change of the reference theory: the skills needed, and especially the nature of the exploration process leading to this change, should be carefully investigated.

We also think that the study we have reported in this paper could be developed further in order to understand better the nature of the *exploration of the situation* described by a given statement and the conditions which allow to make a productive connection between such exploration, *transformational reasoning* and *construction of proof*. The need for further investigations is made clear in the following example.

In an Alessandria University orientation (non selective) test, students had to prove that *"Each number which is even and larger than 2 can be written as the sum of two different odd numbers"*. In spite of the apparent ease of the task, about 90% of students were unable to produce a complete proof.

The most common approaches can be described as follows:

i) after some rather casual numerical trials, some students proved that *"the sum of two different odd numbers is even"*; this approach could be the effect of an effort to *transform* the statement, given the difficulty of proving it, with a final approach to a statement not equivalent to the original one but easier to prove!

ii) some students considered many numerical cases, without finding any regularity; in this case, the *exploration* remained non-goal oriented (as concerns the development of the proving process), although it confirmed the validity of the statement in many numerical cases (so providing arguments for its plausibility);

iii) other students wrote(for instance)4=1+3;6=1+5;8=1+7;10=1+9;12=1+11,but they were not able to elicit the general relation *("each even number is the sum of two odd numbers,1 and the preceding number");* indeed some of them grasped the existence of a *"regularity"*, but wrote:*"I am not able to write it in general"*.In few cases, the interviews recorded after the test revealed that the difficulty derived from insufficient mastery of algebraic language; in other cases, the student was not able to see that the second addendum of the sum was always odd because it preceded an even number!

These behaviours show the necessity of taking into account other aspects of the proving process, which concern both the exploration process and the transformation of the statement: the nature of the "control function" of transformations, and how to develop it (see i); why does the exploration process prove in some cases absolutely blind (see ii); the role (and the difficulty) of that particular exploration which aims to interpret the results of a transformation or a preceding exploration (see iii).

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